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Variational problems arising in classical mechanics and nonlinear elasticity

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VARIATIONAL PROBLEMS ARISING IN CLASSICAL MECHANICS AND NONLINEAR ELASTICITY

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Summary

In this thesis we consider two different classes of variational problems.

First, one-dimensional problems arising from classical mechanics where the problem is to determine whether there is a unique function $\eta_0(x)$ which minimises the energy functional of the form

$$\mathcal{I}(\eta) = \int_a^b L(x, \eta(x), \eta'(x))\ dx.$$

We will investigate uniqueness by making a change of dependent and independent variables and showing that for a class of integrands L with a particular kind of scaling invariance the resulting integrand is completely convex. The change of variables arises by applying results from Lie group theory as applied in the study of differential equations and this work is motivated by [60] and [68].

Second, the problem of minimising energy functionals of the form

$$E(\boldsymbol{u}) = \int_A W(\nabla \boldsymbol{u}(\boldsymbol{x}))\ \mathrm{dx}$$

in the case of a nonlinear elastic body occupying an annular region $A \subset \mathbb{R}^2$ with $\boldsymbol{u} : \bar{A} \to \bar{A}$. This work is motivated by [57] (in particular the example of §4). We will consider rotationally symmetric deformations satisfying prescribed boundary conditions. We will show the existence of minimisers for stored energy functions of the form $W(F) = \tilde{g}(|F|, \det(F))$ in a class of general rotationally symmetric deformations of a compressible annulus and for stored energy functions of the form $W(F) = \bar{g}(|F|)$ in a class of rotationally symmetric deformations of an incompressible annulus. We will also show that in each case the minimisers are solutions of the full equilibrium equations. A model problem will be considered where the energy functional is the Dirichlet integral and it will be shown that the rotationally symmetric solution obtained is a minimiser among admissible non-rotationally symmetric deformations. In the case of an incompressible annulus, we will consider the Dirichlet integral as the energy functional and show that the rotationally symmetric equilibrium solutions in this case are weak local minimisers in a general class of admissible deformations, and that this result can be extended to the case of slightly compressible materials. We also investigate the problem of whether minimisers of a specific energy functional in a class of general deformations are necessarily rotationally symmetric by introducing the notion of an area preserving symmetrisation.

"They say 'Hard work never hurt anyone', but I figure 'Why take the chance ?'"

(*Ronald Reagan (attr.)*)

Trust in the LORD always,
        and lean not on your own understanding.
In all your ways acknowledge Him,
        and He will keep your paths straight.

(*Proverbs 3: 5 - 6*)

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Chapter 1

Introduction.

1.1 Variational problems.

A central problem in the calculus of variations is to find functions which minimise (or maximise) a given integral functional in a given class of functions satisfying prescribed boundary conditions. More precisely, consider the integral functional

$$\mathcal{I}(\boldsymbol{u}) = \int_\Omega L(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\ \mathrm{dx} \tag{1.1.1}$$

where $\Omega \subset \mathbb{R}^n$ is bounded and open and $\mathcal{I}(\boldsymbol{u})$ is defined on maps $\boldsymbol{u} : \Omega \to \mathbb{R}^m$. We associate $\nabla \boldsymbol{u}$ with the $m \times n$ Jacobian matrix of its partial derivatives and so $\nabla \boldsymbol{u}(\boldsymbol{x}) : \Omega \to M^{m\times n}$ where $M^{m\times n}$ is the space of real $m \times n$ matrices (and we write $\nabla \boldsymbol{u}(\boldsymbol{x}) = (u^i_{,j}(\boldsymbol{x})))$. We assume that L is a smooth real-valued function and we study minimisers (or maximisers) of $\mathcal{I}$ in a given set of admissible mappings. The earliest examples of one-dimensional variational problems ($m = n = 1$) arose from isoperimetric problems studied by the ancient Greeks, examples of which are Dido's problem - of finding the maximum area enclosed by a closed curve of fixed length - and a Zenodoros problem - of finding the maximum volume enclosed by a surface of fixed area. The modern era of the calculus of variations began in 1696 with the solution of Johann Bernoulli to the brachistochrone problem - of finding the shape of the curve joining two prescribed points in space along which a bead could slide under gravity without friction in the least time (the solution being the arc of a cycloid). Since the 1970's, variational techniques have been used as a major tool in the field of nonlinear elasticity where higher-dimensional problems ($m > 1$ or $n > 1$) arise, the fundamental problem being to show the existence and determine the properties of deformations which minimise the total energy stored in an elastic body.

1.1.1 One-dimensional variational problems.

Consider the integral functional

$$\mathcal{I}(\eta) = \int_a^b L(x, \eta(x), \eta'(x))\, dx$$

where $\eta : [a, b] \to \mathbb{R}$. We wish to minimise $\mathcal{I}(\eta)$ on the set of admissible functions $\mathcal{A} = \{\eta \in W^{1,1}((a, b)) : \eta(a) = c, \eta(b) = d\}$. Here $W^{1,1}((a, b))$ is the space of absolutely continuous functions with norm

$$\| f \|_{1,1} = \int_a^b |f(x)|\, dx + \int_a^b |f'(x)|\, dx$$

for $f \in W^{1,1}((a, b))$. The integrand L is assumed to be smooth. There are two main approaches to the problem of minimising $\mathcal{I}(\eta)$ on $\mathcal{A}$. The first is due to Weierstrass, and is concerned with determining whether a given function, η_0, is a local minimiser of $\mathcal{I}(\eta)$. We say that $\eta_0 \in W^{1,1}((a, b))$ is a *strong local minimiser* of $\mathcal{I}(\eta)$ if there exists $\varepsilon > 0$ such that $\mathcal{I}(\eta) \geq \mathcal{I}(\eta_0)$ for all η in $\mathcal{A}$ such that $\| \eta - \eta_0 \|_\infty < \varepsilon$ where

$$\| f \|_\infty := \inf\{\alpha : |f(x)| \leq \alpha \text{ a.e. in } (a, b)\},$$

and $\eta_0 \in W^{1,1}((a, b))$ is a *weak local minimiser* of $\mathcal{I}(\eta)$ if there exists $\varepsilon > 0$ such that $\mathcal{I}(\eta) \geq \mathcal{I}(\eta_0)$ for all η in $\mathcal{A}$ such that $\| \eta - \eta_0 \|_{1,\infty} < \varepsilon$ where

$$\| f \|_{1,\infty} := \max\{\| f \|_\infty, \| f' \|_\infty\}.$$

In one-dimensional variational problems necessary conditions for $\eta_0 \in C^1((a, b)) \cap \mathcal{A}$ to be a weak local minimiser of $\mathcal{I}(\eta)$ (and hence also a strong local minimiser) are that the first variation should vanish, that is

$$\delta\mathcal{I}(\eta_0)(\phi) := \frac{d}{d\varepsilon}\mathcal{I}(\eta_0 + \varepsilon\phi)|_{\varepsilon=0} = 0 \tag{1.1.2}$$

for all $\phi \in \{f \in C^1([a, b]) : f(a) = f(b) = 0\} =: C_0^1(a, b)$, and the second variation be positive *semi-definite*, that is

$$\delta^2\mathcal{I}(\eta_0)(\phi, \phi) := \frac{d^2}{d\varepsilon^2}\mathcal{I}(\eta_0 + \varepsilon\phi)|_{\varepsilon=0} \geq 0 \tag{1.1.3}$$

for all $\phi \in C_0^1(a, b)$. A third necessary condition for η_0 to be a strong local minimiser is the *Weierstrass condition*: this states that

$$L(x, \eta_0(x), p) \geq L(x, \eta_0(x), \eta_0'(x)) + (p - \eta_0'(x))L_{,3}(x, \eta_0(x), \eta_0'(x)) \qquad (1.1.4)$$

for all $x \in [a, b]$ and for all $p \in \mathbb{R}$. (We note that convexity of L in the third argument implies the Weierstrass condition (1.1.4).)
Sufficient conditions for η_0 to be a weak local minimiser of $\mathcal{I}(\eta)$ are that the first variation should vanish and the second variation be (uniformly) positive *definite*, that is

$$\delta^2\mathcal{I}(\eta_0)(\phi, \phi) \geq \mu \int_a^b \phi^2(x) + (\phi'(x))^2 \, dx \qquad (1.1.5)$$

for all $\phi \in C_0^1(a, b)$, where $\mu > 0$. However, these are not sufficient for η_0 to be a strong local minimiser. A third sufficient condition required for η_0 to be a strong local minimiser is the *strengthened Weierstrass condition*: this states that for any $x \in [a, b]$ there exists a neighbourhood of points (x, η, η') about $(x, \eta_0(x), \eta_0'(x))$ such that

$$L(x, \eta, p) \geq L(x, \eta, \eta') + (p - \eta')L_{,3}(x, \eta, \eta') \qquad (1.1.6)$$

for all $p \in \mathbb{R}$. Thus there is a neighbourhood of (x, η_0, η_0') such that the tangent to the graph of $L(x, \eta, p)$ at $p = \eta'$ does not lie above the graph. These three conditions together are sufficient to show that η_0 is a strong local minimiser.
However, there is still the question of the existence of absolute minimisers. This was answered by Tonelli, who was able to determine conditions under which $\mathcal{I}(\eta)$ attained an absolute minimum by working with the integral directly. This led to the direct method approach. In the direct method of the calculus of variations the functional is considered directly and existence of minimisers is shown under the conditions that $L(x, \eta, \cdot)$ is convex for each $(x, \eta) \in [a, b] \times \mathbb{R}$ and that L satisfies the coercivity condition $L(x, \eta, p) \geq \Phi(p)$ where $\Phi(p)$ is such that $\lim_{|p| \to \infty} \dfrac{\Phi(p)}{|p|} \to \infty$. Tonelli also presented a partial regularity result: if, in addition to the above conditions, L is such that $L_{,33}(x, \eta, p) > 0$ for each (x, η, p), then a minimiser η_0 of $\mathcal{I}(\eta)$ is C^2 on the interval $[a, b] \setminus \Omega_0$ and satisfies the Euler-Lagrange equation

$$L_{,2}(x, \eta(x), \eta'(x)) - \frac{d}{dx} \{L_{,3}(x, \eta(x), \eta'(x))\} = 0$$

on $[a, b] \setminus \Omega_0$, where Ω_0 is a closed set of Lebesgue measure zero. Ball & Mizel [12] have provided examples where Ω_0 is not empty. Thus it is possible for minimisers to be singular and not C^1.

We briefly discuss the case of vector functions of one variable ($m > 1$, $n = 1$). In this case it can be shown that if $\boldsymbol{\eta}_0'$ is essentially bounded then C^1 and C^2 regularity of $\boldsymbol{\eta}_0$ under conditions of coercivity, positivity and strict convexity of the integrand have been shown (see [20, §2.6] for details). Also $\boldsymbol{\eta}_0$ satisfies

$$\frac{d}{dx}\{L_{,3}(x, \boldsymbol{\eta}(x), \boldsymbol{\eta}'(x))\} = L_{,2}(x, \boldsymbol{\eta}(x), \boldsymbol{\eta}'(x)).$$

Clarke & Vinter [26] have investigated regularity properties of solutions under conditions weaker than those of Tonelli.

There are many texts giving a detailed presentation of the calculus of variations (see, for example, [20], [32]). A brief review of the development and present state of the calculus of variations, which includes the results of Weierstrass and Tonelli, is given by Ball [10].

1.1.2 Nonlinear elasticity.

We now discuss the progress made in nonlinear elasticity. Throughout our work (in Chapters 4 - 7) we will assume that the material is hyperelastic. In hyperelasticity the total stored energy E for an admissible deformation $\boldsymbol{u}$ of a nonlinear elastic material occupying the region Ω in its undeformed state is

$$E(\boldsymbol{u}) = \int_\Omega W(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}))\ \mathrm{dx}$$

where W is a real-valued function known as the stored energy function and $\nabla \boldsymbol{u}$ is commonly referred to as the deformation gradient. The principal problem is that of showing existence of deformations that minimise the total stored energy of a nonlinear elastic body under prescribed boundary conditions and prescribed loads. In this setting significant progress was made in 1977 by John Ball ([6], [7]). In these works he adopted the variational approach that Stuart Antman ([4], [5]) took in showing the existence of equilibrium solutions for various problems involving nonlinear elastic rods and axisymmetric shells. Ball investigated constitutive hypotheses that were physically reasonable and that analytically led to a proof of the existence of minimisers by the direct method of the calculus of variations. He showed that convexity of the stored energy function with respect to the deformation gradient was unrealistic as a general constitutive hypothesis for nonlinear elasticity problems in higher dimensions, and proposed a more realistic alternative, the quasiconvexity condition of Morrey [48]. Under appropriate growth conditions, Morrey showed that quasiconvexity of W in F is necessary and sufficient for E to be sequentially weakly lower semicontinuous, that is

$$E(\bar{\boldsymbol{u}}) \leq \liminf_{n\to\infty} E(\boldsymbol{u}_n)$$

whenever $\boldsymbol{u}_n \rightharpoonup \bar{\boldsymbol{u}}$ in an appropriate Sobolev space (see [48, Theorem 4.4.5] page 117). However, Ball noted that the growth conditions required for Morrey's results were incompatible with nonlinear elasticity, since they did not allow stored energy functions that satisfy the growth condition

$$W(\boldsymbol{x}, F) \to \infty \text{ as } \det(F) \to 0. \tag{1.1.7}$$

This condition corresponds to the requirement of an infinite amount of energy to compress a finite volume to zero volume. In [6] Ball considers polyconvex stored energy functions. These are functions $W(\boldsymbol{x}, F)$ which are expressible as a convex function of F and its minors of all orders. In particular, for the case $m = n = 3$, polyconvex stored energy functions are expressible as $W(\boldsymbol{x}, F) = g(\boldsymbol{x}, F, \operatorname{adj}(F), \det(F))$ where $g(\boldsymbol{x}, \cdot, \cdot, \cdot) : M^{3\times3} \times M^{3\times3} \times (0, \infty) \to \mathbb{R}$ is a convex function on $\mathbb{R}^{19}$ for almost all $\boldsymbol{x} \in \Omega$. Assuming polyconvexity, and under suitable coercivity hypotheses, Ball used the direct method of the calculus of variations to show the existence of minimisers among functions in the Sobolev space $W^{1,p}(\Omega)$ satisfying prescribed boundary conditions and the orientation-preserving condition

$$\det(\nabla \boldsymbol{u}) > 0 \text{ a.e. in } \Omega \tag{1.1.8}$$

([6, Theorem 7.6]). Also he showed the existence of minimisers among functions in $W^{1,p}(\Omega)$ in the case of incompressible elasticity where the deformations satisfy the volume-preserving condition

$$\det(\nabla \boldsymbol{u}) = 1 \text{ a.e. in } \Omega \tag{1.1.9}$$

([6, Theorem 7.8]). See §3.2 for the hypotheses for both cases.

Although Ball establishes existence of minimisers, one question arising from [6] concerns their smoothness, in particular the conditions under which minimisers are smooth enough to satisfy the weak form of the equilibrium equations. If we consider the case of a homogeneous material then the energy is of the form

$$E(\boldsymbol{u}) = \int_\Omega W(\nabla \boldsymbol{u}) \, \mathrm{dx}. \tag{1.1.10}$$

If a minimiser $\bar{\boldsymbol{u}}$ of E given by (1.1.10) is C^2 then by standard arguments it must satisfy the Euler-Lagrange equations

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u})\right] = 0, \; i = 1, \dots, n, \text{ in } \Omega. \tag{1.1.11}$$

It is not necessarily the case that $\bar{\boldsymbol{u}}$ satisfies the Euler-Lagrange equations if $\bar{\boldsymbol{u}}$ is only in $W^{1,p}(\Omega)$. Ball [9] has shown the derivation of other forms of the equilibrium equations by considering "outer variations" of the form $\boldsymbol{u}_\varepsilon(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \varepsilon \boldsymbol{v}(\bar{\boldsymbol{u}}(\boldsymbol{x}))$, where $\boldsymbol{v} \in C^1_0(\bar{\boldsymbol{u}}(\Omega))$, and "inner variations" of the form $\boldsymbol{u}_\varepsilon(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x} + \varepsilon \boldsymbol{v}(\boldsymbol{x}))$, where $\boldsymbol{v} \in C^1_0(\Omega)$. If outer variations are used the corresponding Euler-Lagrange equations are

$$\frac{\partial}{\partial u^\alpha}[T_{i\alpha}(\boldsymbol{u})] = 0, \; i = 1, \dots, n, \text{ in } \bar{\boldsymbol{u}}(\Omega), \tag{1.1.12}$$

where $\boldsymbol{T} = (T_{i\alpha})$ is the Cauchy stress tensor defined on $\bar{\boldsymbol{u}}(\Omega)$ by

$$\boldsymbol{T}(\boldsymbol{u}(\boldsymbol{x})) = \frac{1}{\det(\nabla \boldsymbol{u}(\boldsymbol{x}))}\left(\frac{\partial W}{\partial F}(\nabla \boldsymbol{u}(\boldsymbol{x}))\right)(\nabla \boldsymbol{u}(\boldsymbol{x}))^T.$$

If, instead, inner variations are used the corresponding Euler-Lagrange equations are

$$\frac{\partial}{\partial x^\alpha}\left[W(\nabla \boldsymbol{u})\delta^i_\alpha - u^\beta_{,i}\frac{\partial W}{\partial F^\beta_\alpha}(\nabla \boldsymbol{u})\right] = 0, \; i = 1, \dots, n, \text{ in } \Omega. \tag{1.1.13}$$

(see also [14], [49] and [57] for details).
The question of regularity of the minimisers is in general still to be resolved. In the case of multi-dimensional variational problems (m, $n \geq 2$), a major result is that of Evans [28], which shows regularity of minimisers off a closed set of zero Lebesgue measure for homogeneous quasiconvex stored energy functions $W(F)$ satisfying the uniform quasiconvexity condition, that is for some $\gamma > 0$

$$\int_\Omega W(F + \nabla \boldsymbol{\phi}) \, \mathrm{dx} \geq \int_\Omega W(F) + \gamma(1 + |\nabla \boldsymbol{\phi}|^{p-2})|\nabla \boldsymbol{\phi}|^2 \, \mathrm{dx} \tag{1.1.14}$$

for all $F \in M^{m\times n}$ and $\boldsymbol{\phi} \in C^1_0(\Omega)$, where $W(F)$ satisfies $\left|\frac{\partial^2 W}{\partial F^2}(P)\right| \leq C(1 +$

$|P|^{p-2})$ for all $P \in M^{m\times n}$ and for some constant C, and where $2 \leq p < \infty$. This result was extended to include integrals of the form

$$\mathcal{E}(\boldsymbol{u}) = \int_\Omega W(\boldsymbol{x}, \boldsymbol{u}(\boldsymbol{x}), \nabla \boldsymbol{u}(\boldsymbol{x}))\ \mathrm{dx}$$

where the integrand, along with being quasiconvex, satisfies the growth condition $|F|^p \leq W(\boldsymbol{x}, \boldsymbol{u}, F) \leq C(|F|^2 + 1)^{\frac{p}{2}}$, where $C > 0$ and $p \geq 2$ (see, e.g., [34]). Partial regularity has also been shown for $1 < p < 2$ under the assumption that $|W(F)| \leq C(1 + |F|^p)$ (see [19]). However, as with Morrey's growth conditions, in each of these cases the growth hypotheses imposed are unrealistic for nonlinear elasticity, since they are not compatible with the condition (1.1.7).

One case where the minimisers can be shown to be smooth enough to satisfy a weak form of the equilibrium equations is in the analysis of cavitation in elasticity. Ball [8] considered a class of radial deformations of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \frac{\rho(R)}{R}\boldsymbol{x}$$

on the unit ball $B_1 = \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| < 1\}$ and such that $\boldsymbol{u}(\boldsymbol{x}) = \lambda \boldsymbol{x}$ on ∂B_1, where $\lambda \in (0, \infty)$ and $R = |\boldsymbol{x}|$. The deformation exhibits cavitation if $\rho(0) > 0$. The assumption of radial symmetry reduces the variational problem to a one-dimensional problem in the calculus of variations and, for the case where $n = 3$, the total energy E takes the form

$$E(\boldsymbol{u}) = \int_{B_1} W(\nabla \boldsymbol{u})\ \mathrm{dx} = 4\pi \int_0^1 R^2 \Phi\left(\rho'(R), \frac{\rho(R)}{R}, \frac{\rho(R)}{R}\right) dR =: 4\pi I(\rho). \tag{1.1.15}$$

The total stored energy is presented in this form since it is known that (for $n = 3$) W is frame-indifferent and isotropic if and only if there exists a symmetric function $\Phi : \mathbb{R}^3_{++} \to \mathbb{R}$ satisfying $W(F) = \Phi(v_1, v_2, v_3)$ where $\mathbb{R}^3_{++} = \{(c_1, c_2, c_3) \in \mathbb{R}^3 : c_i > 0,\ i = 1, 2, 3\}$ and v_i are the eigenvalues of $(F^T F)^{\frac{1}{2}}$ and are known as the *principal stretches.* (For full details, see [69].) In the case of radial deformations $v_1 = \rho'(R)$, $v_2 = v_3 = \dfrac{\rho(R)}{R}$.

By using variational techniques, Ball was able to show the existence of non-trivial solutions of the equilibrium equations

$$\frac{d}{dR}\left[R^2 \Phi_{,1}\left(\rho'(R), \frac{\rho(R)}{R}, \frac{\rho(R)}{R}\right)\right] = 2R\Phi_{,2}\left(\rho'(R), \frac{\rho(R)}{R}, \frac{\rho(R)}{R}\right) \tag{1.1.16}$$

corresponding to cavitation by showing that I attains a minimum on $\mathcal{A}_\lambda^{\text{sym}}$ where

$$\mathcal{A}_\lambda^{\text{sym}} = \{\rho \in W^{1,1}((0,1)) : \rho(1) = \lambda, \rho'(R) > 0 \text{ a.e. on } (0,1), \rho(0) \geq 0\} \quad (1.1.17)$$

and by showing that for sufficiently large values of λ any minimiser ρ of I on $\mathcal{A}_\lambda^{\text{sym}}$ satisfies $\rho(0) > 0$. The condition $\rho'(R) > 0$ is a consequence of (1.1.8) since $\det(\nabla \boldsymbol{u}) = \rho' \frac{\rho}{R}$. Ball also showed that, for a class of stored energy functions, the minimisers are sufficiently regular in the interval $(0,1]$ and satisfy (1.1.16). This result was generalised in Sivaloganathan [60] which relaxes the conditions on the stored energy function under which cavitation can occur. In [60] Sivaloganathan then went on to show uniqueness of regular and cavitating equilibrium solutions in the case where the domain is a ball $B_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < 1\}$ and in the case where the domain is a spherical shell $B_1^\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^3 : \varepsilon < |\boldsymbol{x}| < 1\}$ of inner radius $\varepsilon \in (0,1)$. A general existence theory that allowed for the formation of holes with no assumption of radial symmetry was developed by Müller & Spector (see [49]). In our work we will be adopting some of the ideas presented in these works on cavitation. However, the actual problem we will be considering is not a problem directly involving cavitation, and we refer the interested reader to [39] for a summary of work on cavitation.

We now discuss some of the progress made in incompressible elasticity. In incompressible elasticity the deformations $\boldsymbol{u}$ are subject to the constraint $\det(\nabla \boldsymbol{u}) = 1$ almost everywhere. This constraint ensures that all admissible deformations are locally volume-preserving. Ball [6] shows the existence of minimisers in a class of incompressible deformations. For incompressible bodies the laws of force and momentum balance result in the Cauchy stress tensor depending on a pressure term $\bar{P}(\boldsymbol{x})$ for incompressible materials. Such a response has been derived for hyperelasticity by, for example, Rivlin [58]. Le Tallec & Oden [46] have given sufficient conditions for the existence of such a pressure term $\bar{P}(\boldsymbol{x})$ and $\bar{P}(\boldsymbol{x})$ is unique by the arguments of Ball [8]. See [44], [51] and [69] for corresponding results for materials which are not necessarily hyperelastic. For incompressible hyperelastic materials, the equilibrium equations are

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F_\alpha^i}(\boldsymbol{x}, \nabla \boldsymbol{u}) - \bar{P}(\boldsymbol{x})(\operatorname{adj}(\nabla \boldsymbol{u}))_i^\alpha\right] = 0, \; i = 1, \ldots, n.$$

For the incompressible case Fosdick & MacSithigh [30] present necessary conditions for minimisers in terms of the first and second variations. They also derive the Legendre-Hadamard and Weierstrass conditions for the incompressible case.

We next consider deformations of slightly compressible materials, that is materials which only undergo small local changes in volume. In [45], Le Dret models such materials by energy functionals of the form

$$E(\boldsymbol{u}) = \int_\Omega \bar{W}(\nabla \boldsymbol{u}) + \frac{1}{\delta} h(\det(\nabla \boldsymbol{u}))\ \mathrm{dx}$$

where δ is a compressibility parameter, with $\delta = 0$ being the incompressible case, and $\bar{W}(F)$ is the stored energy function for an incompressible material. Le Dret assumes that a solution of the equilibrium equations in the slightly compressible case $\bar{\boldsymbol{u}}_\delta$ can be written as

$$\bar{\boldsymbol{u}}_\delta = \bar{\boldsymbol{u}}_0 + \delta \boldsymbol{w}_1 + o(\delta) \tag{1.1.18}$$

in $W^{2,p}(\Omega)$ and shows then that $\det(\nabla \bar{\boldsymbol{u}}_0) = 1$ and hence that $\bar{\boldsymbol{u}}_0$ is a solution of the equilibrium equations in the incompressible case (see §6.3 for a more detailed discussion). A second representation of the energy functional for slightly compressible materials is given by Ogden [50], and has been considered by Charrier *et al* [21]. In [21], for the case $m = n = 3$, they show, under the assumption that the pressure is related to the change of volumes by $\bar{P}(\boldsymbol{x}) = g(\det(\nabla \boldsymbol{u}))$, where $g(d)$ is such that $g(1) = 0$, that the stored energy function in the case of a slightly compressible material of the form

$$W(F) = W^*\left(\frac{F}{(\det(F))^{\frac{1}{3}}}\right) + \int_1^{\det F} g(v)\ dv \tag{1.1.19}$$

where $W^*(F) = W(F)|_{\det(F)=1}$ is determined by experiment for which it is assumed that volume changes are negligible. Also, on considering a class of functions W^* of the form

$$W^*(F) = \sum_{i=1}^{M} a_i\ \mathrm{tr}((F^T F)^{\frac{\alpha_i}{2}}) + \sum_{i=1}^{N} b_i\ \mathrm{tr}((\mathrm{adj}(F^T F))^{\frac{\beta_i}{2}})$$

with a_i, $b_i > 0$ and α_i, $\beta_i \geq 1$, so that W^* is polyconvex and coercive, and requiring

$$G(y) = \int_1^y g(v) dv$$

to be convex and such that $G(y) \to \infty$ as $y \to 0_+$, $G(y) = 0$ if and only if $y = 1$ and $G(y) \geq C y^\gamma$ for some $C > 0$ and $\gamma > 1$ with y sufficiently large, they show that $W(F)$ as given by (1.1.19) is polyconvex and coercive. It is also shown that as the compressibility tends to zero the equilibrium solutions and the energy functional

$$E_\varepsilon(\boldsymbol{u}) = \int_\Omega W^*\left(\frac{\nabla \boldsymbol{u}}{(\det(\nabla \boldsymbol{u}))^{\frac{1}{3}}}\right)\,\mathrm{dx} + \frac{1}{\varepsilon}\int_\Omega \left\{\int_1^{\det(\nabla \boldsymbol{u})} g(v)\,dv\right\}\,\mathrm{dx}$$

converge to those for the incompressible case ([21, Theorem 11]).
We note that in the incompressible case local stability of minimisers (in the sense of positive definiteness of the second variation) has also been investigated in cases of uniaxial extension and rotations about distinct axes of an elastic layer (for example in [22], [23] & [37]), but these results are in a restrictive class of semi-inverse deformations.
We briefly mention the question of uniqueness of solutions to the problem of minimising the total energy of the form (1.1.10). It was shown by Hill [38] that strict convexity of W with respect to $\nabla \boldsymbol{u}$ implies uniqueness of minimisers (but, as already mentioned, convexity is an unrealistic constitutive assumption for nonlinear elasticity in multi-dimensional problems). Knops & Stuart [42] were able to show uniqueness of solutions for star-shaped domains under the more realistic assumption of quasiconvexity of the integrand. John [41] presented the example of a nonlinear elastic body occupying an annular region $A = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < b\}$ and deformations $\boldsymbol{u}$ satisfying identity boundary conditions ($\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ on ∂A) as a heuristic example of non-uniqueness for non-star-shaped domains. The intuitive reason why multiple equilibria are expected is that rotation of one of the boundaries by an integer multiple of 2π results in different equilibria satisfying the same boundary conditions (see Figure 1.1). This heuristic example was the basis of recent work by Post & Sivaloganathan [57] in which they show the existence of infinitely many local minimisers for displacement boundary-value problems on a two-dimensional annular domain and a three-dimensional tubular domain with annular cross-section. In the two-dimensional case they also show the existence of minimisers in a set of rotationally symmetric deformations $\boldsymbol{u}$ of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \rho(R)\begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix}, \qquad (1.1.20)$$

where $R = |\boldsymbol{x}| \in [a, b]$, $\theta \in [0, 2\pi)$, and where $\rho : [a, b] \to [a, b]$ represents the radial stretch and $\psi : [a, b] \to \mathbb{R}$ represents the angle of twist, for stored energy functions

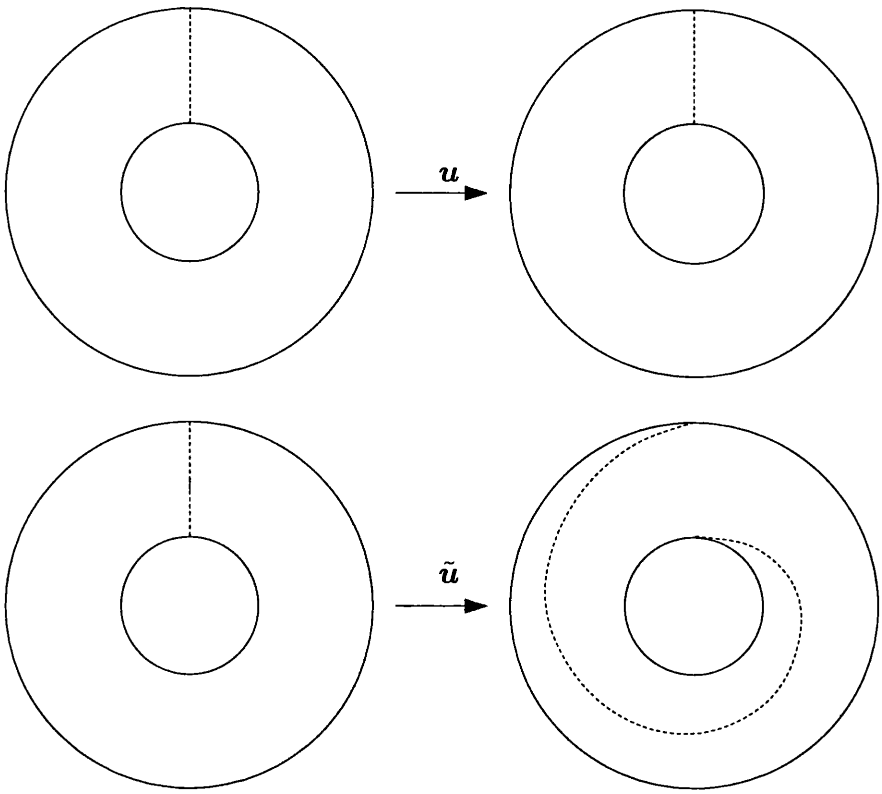

Figure 1.1: Two deformations satisfying $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ on ∂A.

of the form

$$W(F) = \frac{1}{2}|F|^2 + h(\det(F)), \tag{1.1.21}$$

where h is C^2, convex and such that $h(d) \to \infty$ as $d \to 0, \infty$, so that (1.1.7) holds. In considering rotationally symmetric deformations the problem is again reduced to a one-dimensional variational problem. However, here the stored energy function contains two functions $(\rho(R), \psi(R))$, and thus the energy functional is an integral of the form

$$E(\boldsymbol{u}) = 2\pi I(\rho, \psi) := 2\pi \int_a^b R\left[\frac{1}{2}\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\} + h\left(\rho'\frac{\rho}{R}\right)\right] dR. \tag{1.1.22}$$

The existence of a minimiser, $(\hat{\rho}, \hat{\psi})$ say, in the set

$$\mathcal{A}_N^{\text{sym}} = \{(\rho, \psi) \in W^{1,1}((a,b)) : \rho(a) = a, \rho(b) = b, \rho'(R) > 0 \text{ a.e. on } (a,b), \\ \psi(a) = 0, \psi(b) = 2N\pi\} \tag{1.1.23}$$

(where $N \in \mathbb{N} \cup \{0\}$ is fixed) is shown by the direct method of the calculus of variations. It is noted in [57] that $(\hat{\rho}, \hat{\psi})$ is C^2, satisfies the rotationally symmetric Euler-Lagrange equations

$$\begin{aligned} \frac{d}{dR}\left[R\rho'(R) + \rho(R)h\left(\rho'(R)\frac{\rho(R)}{R}\right)\right] \\ = \frac{\rho(R)}{R} + R\rho(R)(\psi'(R))^2 + \rho'(R)h\left(\rho'(R)\frac{\rho(R)}{R}\right) \end{aligned} \tag{1.1.24}$$

and

$$\frac{d}{dR}\left[R\rho^2(R)\psi'(R)\right] = 0 \tag{1.1.25}$$

and gives rise to a solution $\hat{\boldsymbol{u}}$ of the full equilibrium equations (1.1.11). A related problem is studied in Chen & Rajagopal [24] in which existence and uniqueness of equilibrium solutions are shown for a very specific class of circumferential shear deformations $(R, \Theta, Z) \to (r, \theta, z)$ of the form $r = R$, $\theta = \Theta + f(R)$, $z = Z$ of an incompressible three-dimensional cylindrical body with annular cross-section, where $\Omega = \{(R, \Theta, Z) : a < R < b,\ 0 < \Theta < 2\pi,\ -\infty < Z < \infty\}$ and f is such that $f(a) = \phi$ (where $\phi \in [0, 2\pi)$ is fixed), $f(b) = 0$ (see also [67]). The approaches of [24] and [67], however, are not variational.

1.2 Outline of the thesis.

This thesis will consider two different classes of variational problems and is correspondingly split into two parts.

1.2.1 Symmetries occurring in one-dimensional variational boundary value problems.

In Chapter 2 we will consider classical examples of one-dimensional variational boundary value problems. The examples include those considered by Troutman in [68]:

1. the problem of finding the curve along which a bead would slide under gravity without friction in the least time (commonly referred to as the brachistochrone problem);
2. the problem of finding the maximum area enclosed by a closed curve of fixed length (Dido's problem);
3. the problem of finding the maximum volume enclosed by a surface of revolution of fixed area (a Zenodoros problem).

In each of these cases there is a unique minimiser. In [68] Troutman demonstrates this uniqueness by showing that an ad hoc change of dependent variable results in a strictly convex integrand, which then leads to a proof of uniqueness. We will investigate a systematic method for determining a change of dependent and independent variables for which the resultant integrand is convex. We adopt a different approach to that in [68] by using the theory of Lie groups and their application to differential equations. This approach is motivated by a proof in [60] which shows uniqueness in an example of cavitation in an elastic sphere. In this chapter we will derive a systematic method for generating a change of dependent and independent variables similar to that made in [60] where the method is dependent on the symmetry properties of the integral $\mathcal{I}(\eta)$ (given by (1.1.1)) and the corresponding Euler-Lagrange equation under a given group of scaling transformations. We will show that the integrand in these new variables is convex and we will determine the conditions under which uniqueness can be shown.

1.2.2 Variational problems for deformations of an annulus.

The second part of this thesis concentrates on variational problems arising in nonlinear elasticity. In Chapter 3 we will give the basic definitions and results from functional analysis that will be useful in nonlinear elasticity, and there will be a basic introduction to nonlinear elasticity.

In Chapters 4 - 7 we will consider variational problems of a two-dimensional annulus satisfying prescribed displacement boundary conditions. This work is based on the example in §4 of Post & Sivaloganathan [57], in which they show existence of multiple equilibria for the pure displacement boundary-value problem of minimising the total energy of a compressible nonlinear elastic annulus in a class of rotationally symmetric deformations $\boldsymbol{u}$ of the form (1.1.20) for stored energy functions of the form (1.1.21). The deformations map the annulus to itself and equal the identity on the boundary.

Existence and regularity of minimisers.

In Chapter 4 we continue to consider rotationally symmetric deformations of the form (1.1.20) of a compressible annulus. In §4.1 it will be shown that the existence result of [57, §4], which is for polyconvex stored energy functions of the form (1.1.21), can be extended to a more general class of polyconvex stored energy functions of the form $W(F) = \tilde{g}(|F|, \det(F))$ where $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ is convex, C^2, satisfies $\tilde{g}_{,1} > 0$ and $\tilde{g} \to \infty$ as $\det(F) \to 0, \infty$ (Proposition 4.1.2). We will also show, under additional growth conditions on $\tilde{g}$, that any minimiser $(\hat{\rho}, \hat{\psi}) \in \mathcal{A}_N^{\mathrm{sym}}$ (given by (1.1.23)) is C^2 on (a, b), satisfies the rotationally symmetric Euler-Lagrange equations (Proposition 4.1.3), and gives rise to a solution $\hat{\boldsymbol{u}}$ of the full equilibrium equations (1.1.11) (with $n = 2$) (Proposition 4.1.5). These results are stated without proof in [57] for stored energy functions of the form (1.1.21).

An explicit example: the degenerate case.

In general it is impossible to solve the Euler-Lagrange equations (1.1.24) and (1.1.25) in order to obtain an explicit solution even if h is known. Hence, we will consider a simpler case where the energy functional is reduced to the Dirichlet integral. In §4.2 we will consider a degenerate example where $W(F) = \dfrac{1}{2}|F|^2$, obtained by putting $h \equiv 0$ into (1.1.21): we call this degenerate as the stored energy function no longer satisfies $W(F) \to \infty$ as $\det(F) \to 0$. As we are considering the Dirichlet integral as the total energy functional, and as $W(F)$ is strictly convex in F, in order to gain existence of minimisers among deformations that allow twisting of the annulus we must allow $\det(\nabla \boldsymbol{u}) \geq 0$ which will result in deformations that map regions of positive area to regions of zero area, such as deformations that map part of the annulus to the inner boundary (see Figure 1.2). As the Euler-Lagrange equations (the vector Laplace equation) can be solved to give an explicit solution (Lemma 4.2.2), an explicit form of a possible candidate for a minimiser is obtained (Proposition 4.2.1). This notion of degenerate minimisers has been considered by Sivaloganathan in [64] in the context of cavitation of the unit ball.

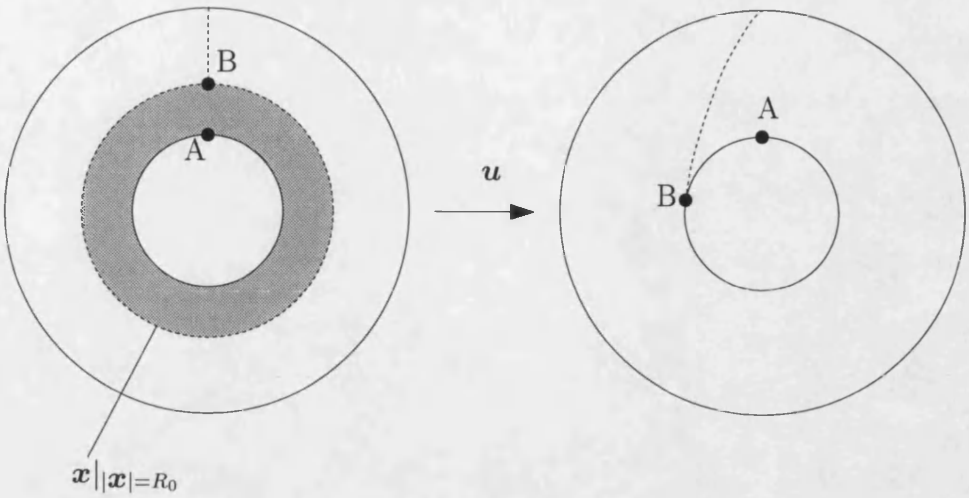

Figure 1.2: The degenerate case: a possible deformation $\boldsymbol{u}$.

The incompressible case.

In Chapter 5 we will consider the problem of minimising the total energy E for stored energy functions W of the form $W(F) = \bar{g}(|F|)$ in the class of rotationally symmetric deformations of the form (1.1.20) of a incompressible annulus. Here, the deformations considered $\boldsymbol{u}$ are locally volume-preserving, that is $\det(\nabla \boldsymbol{u}) = 1$ a.e., from which it follows that $\rho(R) \equiv R$. Existence, uniqueness and regularity of minimisers is straightforward to prove by the arguments of §4.1. It will be shown that there exists a unique hydrostatic pressure term $\bar{P}(\boldsymbol{x})$ such that a solution to the rotationally symmetric Euler-Lagrange equations gives rise to a solution of the full equilibrium equations (Proposition 5.0.2). As an example (Example 5.0.3) we will consider the case where the energy functional is the Dirichlet integral. In this case an explicit form of the equilibrium solutions can be obtained.

Minimising properties of rotationally symmetric equilibrium solutions in the degenerate and incompressible cases.

In Chapter 6 we consider the minimising properties of the explicit rotationally symmetric equilibrium solutions obtained in the degenerate compressible case (§4.2) and in the incompressible case (Example 5.0.3) for the case of the Dirichlet integral. In the degenerate compressible case it will be shown that the radial candidate obtained in §4.2 minimises the energy functional in a class of not necessarily rotationally symmetric local deformations (Proposition 6.1.4). In the incompressible case it will be shown that the rotationally symmetric equilibrium solutions obtained in Example 5.0.3 are weak local minimisers in a more general class of not necessarily rotationally symmetric incompressible deformations using the approach in Fosdick & MacSithigh [30]. We will show that the second variation around the explicit symmetric solution $\bar{\boldsymbol{u}}$ can be written in the form

$$\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = \int_A |\nabla\boldsymbol{\varphi}|^2 - 2\bar{P}(\boldsymbol{x}) \det(\nabla\boldsymbol{\varphi}) \text{ dx} \tag{1.2.1}$$

for all $\boldsymbol{\varphi} \in C_0^1(A)$ satisfying $(\text{adj}(\nabla\bar{\boldsymbol{u}}))^T \cdot \nabla\boldsymbol{\varphi} = 0$ where $\bar{P}(\boldsymbol{x})$ is the corresponding hydrostatic pressure (by Proposition 6.2.5). It can also be shown that (1.2.1) is positive definite (Proposition 6.2.9).

Minimising properties of rotationally symmetric equilibrium solutions in the slightly compressible case.

In §6.3 we will investigate minimising properties of equilibrium solutions in a class of general deformations of a slightly compressible annulus under prescribed displacement boundary conditions in the case where the stored energy function is polyconvex. Mathematically the total stored energy is written in the form

$$E_\delta(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla\boldsymbol{u}|^2 + \frac{1}{\delta}h(\det(\nabla\boldsymbol{u})) \text{ dx}$$

where δ is a parameter of compressibility, $\delta = 0$ corresponding to the incompressible case provided that $h(1) = 0$ (with the energy functional being the Dirichlet integral in the incompressible case). Here, the second variation of E_δ around an equilibrium solution $\bar{\boldsymbol{u}}_\delta$ is of the form

$$D^2 E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = \int_A |\nabla\boldsymbol{\varphi}|^2 + \frac{1}{\delta}h''(\det(\nabla\bar{\boldsymbol{u}}))((\text{adj}(\nabla\bar{\boldsymbol{u}}))^T . \nabla\boldsymbol{\varphi})^2 \\ + \frac{2}{\delta}h'(\det(\nabla\bar{\boldsymbol{u}})) \det(\nabla\boldsymbol{\varphi}) \text{ dx}.$$

We assume that an equilibrium solution in the slightly compressible case $\bar{\boldsymbol{u}}_\delta$ converges to an equilibrium solution in the incompressible case $\bar{\boldsymbol{u}}_0$ as the compressibility δ tends to zero, and that $\bar{\boldsymbol{u}}_0$ is such that

$$\delta^2 E_0(\bar{\boldsymbol{u}}_0)(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = \int_A |\nabla\boldsymbol{\varphi}|^2 - 2\bar{P}(\boldsymbol{x}) \det(\nabla\boldsymbol{\varphi}) \text{ dx} > 0$$

for $\boldsymbol{\varphi} \in C_0^1$ satisfying $\text{adj}(\nabla\bar{\boldsymbol{u}})^T \cdot \nabla\boldsymbol{\varphi} = 0$.
In the spirit of Le Dret [45] we will also assume that $\bar{\boldsymbol{u}}_\delta$ can be written as an expansion of the form $\bar{\boldsymbol{u}}_\delta = \bar{\boldsymbol{u}}_0 + \delta\bar{\boldsymbol{u}}_1 + O(\delta^2)$ in $C^2(\bar{A})$ and that $h(1) = h'(1) = 0$.

We will show, under these assumptions, that there exists $\bar{\delta} > 0$ such that for any $\delta \in (0, \bar{\delta})$ the equilibrium solutions $\bar{\boldsymbol{u}}_\delta$ are weak local minimisers among slightly compressible deformations in the sense that $D^2 E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\varphi}, \boldsymbol{\varphi}) > C_\delta \parallel \boldsymbol{\varphi} \parallel_{1,2}^2$ for $\boldsymbol{\varphi} \in W_0^{1,2}(A)$ (see Proposition 6.3.2).

Symmetrisation arguments for deformations of a compressible annulus.

In Chapter 7 we will investigate minimising properties of rotationally symmetric deformations among general deformations. The approach that we will consider is a symmetrisation argument in the spirit of Pólya & Szegö [56]. The idea is that we write the general deformations in the form

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \tilde{\rho}(R, \theta) \begin{pmatrix} \cos(\theta + \tilde{\psi}(R, \theta)) \\ \sin(\theta + \tilde{\psi}(R, \theta)) \end{pmatrix} \tag{1.2.2}$$

(where $\tilde{\rho}(R, \theta)$ corresponds to a radial stretch and $\tilde{\psi}(R, \theta)$ corresponds to the angle of twist). We associate with each $\tilde{\boldsymbol{u}}$ a rotationally symmetric deformation $\boldsymbol{u}$ of the form (1.1.20) and we choose a suitable symmetrisation from $\tilde{\boldsymbol{u}}$ to $\boldsymbol{u}$ in order to show that the energy is lowered. The symmetrisation we will consider in Chapter 7 is an area-preserving symmetrisation. Partial results are presented in two specific cases - the first being where the radial stretch is symmetric (and thus $\tilde{\rho}(R, \theta) \equiv \rho(R)$ - Proposition 7.3.1) and the second being where the angle of twist is symmetric (and thus $\tilde{\psi}(R, \theta) \equiv \psi(R)$ - Proposition 7.3.7).
We conclude in Chapter 8 with a brief discussion of some open problems related to and following from the work in this thesis.

Chapter 2

Symmetries in one-dimensional variational problems.

In this chapter we consider the problem of minimising integral functionals of the form

$$\mathcal{I}(\eta) = \int_a^b L(x, \eta(x), \eta'(x))dx, \tag{2.0.1}$$

on the set

$$\mathcal{A} = \{\eta \in C^2((a,b)) \cup C^0([a,b]) : \eta(a) = \hat{a}, \eta(b) = \hat{b}, \eta(x) > 0\}$$

where $L : [a,b] \times I_2 \times I_3 \to \mathbb{R}$ is C^2 and where $[a,b]$, I_2, $I_3 \subset \mathbb{R}$ are intervals. Without loss of generality we will assume throughout this chapter that $a = 0$, $\eta(0) = 0$, $\eta(b) = \hat{b}$ and that:

(B1) $\eta(x) \in \mathcal{A}$ and is a minimiser of $\mathcal{I}$ on $\mathcal{A}$ (and so $\eta(x)$ is a solution of the Euler-Lagrange equation);

(B2) $L_{,33}(x, \eta, \cdot) > 0$ (so that $L(x, \eta, \cdot)$ is strictly convex in η' for each $(x, \eta) \in (0,b) \times I_2$).

The Euler-Lagrange equation for $\mathcal{I}(\eta)$ is

$$\frac{\partial L}{\partial \eta} - \frac{d}{dx}\left(\frac{\partial L}{\partial \eta'}\right) = 0. \tag{2.0.2}$$

We call $\eta \in C^2((a,b))$ an **extremal** if η satisfies (2.0.2) on (a,b).
We now give the notion of a convex function.

Definition 2.0.1 *Let V be a real vector space and let $f : V \to \mathbb{R}$. Then f is*

convex *if and only if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \tag{2.0.3}$$

*for all x, $y \in V$ and $\lambda \in [0,1]$. Further, f is **strictly convex** if and only if for all $\lambda \in (0,1)$, (2.0.3) holds with equality if and only if $x = y$.*

We wish to investigate whether there is a unique minimiser for these type of variational problems satisfying the above assumptions, and whether there is a systematic method for showing this. This work is motivated by a preprint by Troutman [68] in which he considers integral functionals $\mathcal{I}(\eta)$ of the form

$$\mathcal{I}(\eta) = \int_0^b L(\eta(x), \eta'(x))\, dx$$

for three classical problems. Troutman shows that by replacing η by a new suitably chosen dependent variable g the new integrand $F(g, g')$ formed by this change of variable is a strictly convex function in (g, g') on its domain $D \subset \mathbb{R}^2$. The three classical problems are:

(I) the brachistochrone problem: this is the problem of finding the curve joining two prescribed points in space by which a bead would slide under gravity without friction in the least time. The integral to be minimised is

$$\mathcal{I}(\eta) = \int_0^b \frac{\sqrt{(1 + (\eta'(x))^2)}}{\sqrt{2\eta(x)}}\, dx$$

on the set $\mathcal{A} = \{\eta \in C^1([0,b]) : \eta(0) = 0, \eta(b) = \hat{b}, \eta(x) > 0 \text{ on } (a,b)\}$. If η is replaced by $g = 2\eta^{\frac{1}{2}}$ then the resulting integrand is strictly convex in (g, g')

η is replaced by $g = 2\eta^{\frac{1}{2}}$ then the resulting integrand is strictly convex in

(II) Dido (g, g').
by a closed curve of fixed length. The integral to be minimised is

$$\mathcal{I}(\eta) = -\int_0^b \eta(x)\sqrt{(1 - (\eta'(x))^2)}\, dx$$

on the set $\mathcal{A} = \{\eta \in C^1([0,b]) : \eta(0) = 0, \eta(b) = 0, \eta(x) > 0 \text{ on } (a,b)\}$. If η is replaced by $g = \dfrac{\eta^2}{2}$ then the resulting integrand is strictly convex in (g, g').

(III) a Zenodoros problem: this is the problem of finding the maximum volume enclosed by a surface of revolution of fixed area. The integral to be minimised is

$$\mathcal{I}(\eta) = \pi \int_0^b \eta(x)\sqrt{(1 - (\eta(x)\eta'(x))^2)}\ dx$$

on the set $\mathcal{A} = \{\eta \in C^1([0,b]) : \eta(0) = 0, \eta(b) = 0, \eta(x) > 0 \text{ on } (a,b)\}$. If η is again replaced by $g = \frac{\eta^2}{2}$ then the resulting integrand is strictly convex in (g, g').

The last two are examples of isoperimetric problems. See [68, §3, §4 & §6]. We now give a property of strict convexity that will allow us to show uniqueness.

Theorem 2.0.2 *Let $F : Q \to \mathbb{R}$ be a function defined and differentiable on an open convex subset $Q \subset \mathbb{R}^2$. Then F is convex if and only if for all $\varrho = (\varrho_1, \varrho_2)$, $\varphi = (\varphi_1, \varphi_2) \in Q$,*

$$F(\varrho_1, \varrho_2) \geq F(\varphi_1, \varphi_2) + F_{,1}(\varphi_1, \varphi_2)(\varrho_1 - \varphi_1) + F_{,2}(\varphi_1, \varphi_2)(\varrho_2 - \varphi_2), \qquad (2.0.4)$$

and F is strictly convex provided equality holds in (2.0.4) if and only if $\varrho_1 = \varphi_1$ and $\varrho_2 = \varphi_2$.

(See [25] pp 165-166 or [68].) A consequence of strict convexity lies in the following standard type of uniqueness result.

Theorem 2.0.3 ([68] page 4) *Suppose $F = F(g, g')$ is C^2 and strictly convex on a convex domain $D \subset \mathbb{R}^2$. Let $g_0 \in C^2((c,d)) \cup C^1([c,d])$ be a solution of the Euler-Lagrange equation*

$$\frac{d}{dy}\left(\frac{\partial F}{\partial g'}(g(y), g'(y))\right) = \frac{\partial F}{\partial g}(g(y), g'(y)),\ c < y < d \qquad (2.0.5)$$

such that $F_{,2}(g_0(.), g_0'(.))$ is bounded. Then g_0 minimises $\mathcal{K}(g)$ defined by

$$\mathcal{K}(g) = \int_c^d F(g(y), g'(y))dy \qquad (2.0.6)$$

uniquely on $\mathcal{B} = \{g \in C^1([c,d]) : g(c) = g_0(c), g(d) = g_0(d)\}$.

Proof: For $g \in \mathcal{B}$, let $k = g - g_0$. Then by (2.0.4), noting that equality occurs if and only if $g = g_0$ and $g' = g_0'$, we obtain

$$\begin{aligned}\mathcal{K}(g) - \mathcal{K}(g_0) &= \int_c^d F(g(y), g'(y)) - F(g_0(y), g_0'(y)) \, dy \\ &\geq \int_c^d F_{,1}(g_0(y), g_0'(y))k(y) + F_{,2}(g_0(y), g_0'(y))k'(y) \, dy.\end{aligned} \tag{2.0.7}$$

Now by (2.0.5) we can write

$$\begin{aligned}&\int_c^d F_{,1}(g_0(y), g_0'(y))k(y) + F_{,2}(g_0(y), g_0'(y))k'(y) \, dy \\ &= \int_c^d F_{,1}(g_0(y), g_0'(y))k(y) - \frac{d}{dy}\{F_{,2}(g_0(y), g_0'(y))\}k(y) \, dy \\ &\quad + \lim_{y \to d}[F_{,2}(g_0(y), g_0'(y))k(y)] - \lim_{y \to c}[F_{,2}(g_0(y), g_0'(y))k(y)] = 0,\end{aligned}$$

since $F_{,2}(g_0(y), g_0'(y))$ is bounded and k is continuous and vanishes at both c and d. Hence, $\mathcal{K}(g) \geq \mathcal{K}(g_0)$.
If $g \not\equiv g_0$ then (2.0.7) holds with strict inequality, so that $\mathcal{K}(g) > \mathcal{K}(g_0)$. □

Example 2.0.4 ([68, §4]) *In Dido's problem the integrand is*

$$L(\eta, \eta') = \eta\sqrt{1 - (\eta')^2},$$

which is not convex in (η, η'). However, if we replace η by $g = \frac{1}{2}\eta^2$ the corresponding integrand becomes

$$F(g, g') = -\sqrt{2g - (g')^2},$$

which is strictly convex on the convex set $D = \{(g, g') \in \mathbb{R}^2 : 2g > (g')^2\}$.

However, the examples (I)-(III) (as considered in [68]) offer no systematic method or explanation for deriving the change of variables required to convexify the integrand, and indeed an example is given in which the author is unable to find a change of dependent variable. The approach of this chapter in deriving a systematic method is motivated by the following result in [60] which arises from a problem in nonlinear elasticity.

Example 2.0.5 *In [60] the functional considered is*

$$\mathcal{I}(\eta) = \int_0^1 x^2 \tilde{L}\left(\eta'(x), \frac{\eta(x)}{x}, \frac{\eta(x)}{x}\right) dx,$$

where $\tilde{L}$ is a symmetric function (in the second and third variables) which is convex in η'. We suppose that $\eta \in C^2((0,1])$ is a solution of the Euler-Lagrange equation

$$\frac{d}{dx}\left(x^2\tilde{L}_{,1}\left(\eta'(x), \frac{\eta(x)}{x}, \frac{\eta(x)}{x}\right)\right) = 2x\tilde{L}_{,2}\left(\eta'(x), \frac{\eta(x)}{x}, \frac{\eta(x)}{x}\right)$$

for which $\eta(1) = \lambda > 0$, $\eta(0) > 0$, $\eta'(x) > 0$ for $x \in (0,1]$ and the natural boundary condition $\lim_{x\to 0} x^2\tilde{L}_{,1} = 0$ holds. Under the change of variables

$$y = \frac{\eta}{x} \qquad \textit{and} \qquad g = x^3$$

the solution $\eta(x)$ gives rise to a new function $g(y)$ and the integral becomes

$$\begin{aligned}\mathcal{I}(\eta) = \int_0^1 x^2\tilde{L}\left(\eta'(x), \frac{\eta(x)}{x}, \frac{\eta(x)}{x}\right) dx \\ = \int_\lambda^\infty -\frac{g'(y)}{3}\tilde{L}\left(\frac{3g(y)}{g'(y)} + y, y, y\right) dy = \mathcal{K}(g).\end{aligned}$$

(see [60, Proposition 3.5]). Moreover, the new integrand

$$G(y,g,g') = -\frac{g'}{3}L\left(\frac{3g}{g'} + y, y, y\right)$$

is convex on the convex set

$$D = \left\{(g,g') \in \mathbb{R}^2; g \in (0,\infty), g' \in (-\infty, 0), \frac{3g}{y} + g' \leq 0\right\}$$

for each $y \in (0,\infty)$ (see [60, Proposition 3.7]), and hence it can be shown that there exists at most one equilibrium solution satisfying the given properties and the required boundary conditions (see [60, Theorem 3.8]).

Our aim in this chapter is to show that for certain extremal problems we can generate a change of variables similar to that considered in Example 2.0.5 by a systematic method that is dependent on symmetry properties of $\mathcal{I}(\eta)$ and the Euler-Lagrange equation under a given group of scaling transformations. With this change of variables we will show convexity of the integrand in these new variables and investigate uniqueness of the solutions.
The approach we will take in the rest of this chapter is the following: §2.1 will consist of the required background theory. We will give the definitions of a Lie group and a group of transformations, define the notion of invariance and

show how invariants of a group of transformations can be found. Also, we will discuss invariance properties of ordinary differential equations, with a view to determining the most general form of invariant ordinary differential equations. We will also discuss the idea of a change of coordinates. §2.2 will consider the application of the theory presented in §2.1 to minimisation problems. We will discuss invariance properties of the integral $\mathcal{I}(\eta)$ and the conditions required of the integrand L so that L can be written in a form similar to that in Example 2.0.5. We will relate the invariance properties of $\mathcal{I}(\eta)$ to invariance properties of the Euler-Lagrange equation. We will then show convexity of the integrand, under certain hypotheses, by using the change of variables that was derived in §2.1, and investigate uniqueness of the solution.

2.1 Lie groups and symmetries of ordinary differential equations.

2.1.1 Lie groups.

We first give the definitions of a Lie group and a transformation group acting on a manifold M, where $M \subset \mathbb{R}^m$. (For the definition of an m-dimensional manifold in $\mathbb{R}^m$ see, e.g., [52, Definition 1.1] or [53, Definition 1.1].)

Definition 2.1.1 *A **Lie group** is a group consisting of a set G together with a group action m, such that the set G has the structure of a manifold and for group elements g_1, $g_2 \in G$ the group action $m : G \times G \to G$, $m(g_1, g_2) = g_1 g_2$, and the inversion $i : G \to G$, $i(g_1) = g_1^{-1}$ define smooth (as in infinitely differentiable) maps.*

(See, for example, [52, Definition 1.16] page 15 or [53, Definition 2.1] page 22.)

Definition 2.1.2 ([53, Definition 2.7] page 35) *A **transformation group acting on a manifold** M is determined by a Lie group and a smooth map $\Pi : G \times M \to M$ denoted by $\Pi(g, s)$, $g \in G$, $s \in M$, satisfying*

(1) $\Pi(e, s) = s$, where e is the identity element of G;

(2) $\Pi(g_1, \Pi(g_2, s)) = \Pi(g_1 g_2, s)$, where g_1, $g_2 \in G$.

Example 2.1.3 *An example of a transformation group is the scaling group consisting of $g_\lambda : (x, y) \to (\lambda x, \lambda^\beta y)$, $\lambda > 0$, on $M = \mathbb{R}^2$. Here, $G = \mathbb{R}_+$ and the* *group action is multiplication. Thus*

$$\Pi(g_\lambda, (x, y)) = (\lambda x, \lambda^\beta y)$$

and the identity map occurs when $\lambda = 1$.

It is possible that the group may act only locally, meaning that the group of transformations may not be defined for all points on the manifold or for all elements of the transformation group, but only those sufficiently close to the identity element e. With this in mind, we now define a local group of transformations. (For the definition of a local Lie group see, e.g., [52, Definition 1.20].)

Definition 2.1.4 ([52, Definition 1.25], page 21) *A **local group of transformations acting on a manifold** M is given by a (local) Lie group, an open subset $\mathcal{U}$, with $\{e\} \times M \subset \mathcal{U} \subset G \times M$, which is the domain of definition of the group action m, and a smooth map $\Pi : \mathcal{U} \to M$ with the following properties:*

(L1) If $(g_2, s) \in \mathcal{U}$, $(g_1, \Pi(g_2, s)) \in \mathcal{U}$ and $(g_1 g_2, s) \in \mathcal{U}$, then $\Pi(g_1, \Pi(g_2, s)) = \Pi(g_1 g_2, s)$.

(L2) For all $s \in M$, $\Pi(e, s) = s$.

(L3) If $(g_1, s) \in \mathcal{U}$, then $(g_1^{-1}, \Pi(g_1, s)) \in \mathcal{U}$ and $\Pi(g_1^{-1}, \Pi(g_1, s)) = s$.

(Note that, except for the assumption of the form of the domain $\mathcal{U}$, (L3) follows directly from (L1) and (L2).)

Definition 2.1.5 *A **one-parameter group of transformations** is a local group of transformations acting on a manifold M where the set G is $\mathbb{R}$ and the operation m is addition. Thus $\Pi = \Pi(\varepsilon, s)$, $s \in M$, where ε is a continuous real-valued parameter, with $\varepsilon = 0$ corresponding to the identity element, and the group action $g_1 g_2 = \varepsilon_1 + \varepsilon_2$ where ε_1 corresponds to g_1 and ε_2 corresponds to g_2.*

Example 2.1.6 *An example of a one-parameter group of transformations acting on $M = \mathbb{R}^3$ is a group of scalings where $\Pi(\varepsilon, (x, y, z)) = (e^\varepsilon x, e^{\beta\varepsilon} y, e^{(\beta-1)\varepsilon} z)$. Here, the identity element is $\varepsilon = 0$ and it is clear that $\Pi(\varepsilon_1 + \varepsilon_2, (x, y, z)) = (e^{\varepsilon_1+\varepsilon_2} x, e^{\beta(\varepsilon_1+\varepsilon_2)} y, e^{(\beta-1)(\varepsilon_1+\varepsilon_2)} z) = \Pi(\varepsilon_1, \Pi(\varepsilon_2, (x, y, z)))$.*

Example 2.1.7 *A second example of a one-parameter group of transformations acting on $M = \mathbb{R}^3$ is a group of translations where $\Pi(\varepsilon, (x, y, z)) = (x, y + \varepsilon, z)$. It is clear that $\Pi(\varepsilon_1 + \varepsilon_2, (x, y, z)) = (x, y + \varepsilon_1 + \varepsilon_2, z) = \Pi(\varepsilon_1, \Pi(\varepsilon_2, (x, y, z)))$.*

As indicated in [52], we will make the assumption that all manifolds and Lie groups are **connected**. This restricts our attention to group transformations like rotations which can be continuously connected to the identity element of the group, and excludes discrete group transformations such as reflections. We also note that a group of transformations acting on a manifold M is **connected** if

(C1) G is a connected set and M is a connected manifold;

(C2) $\mathcal{U} \subset G \times M$ is a connected open set;

(C3) for each $x \in M$, $G_x := \{g \in G : (g, x) \in \mathcal{U}\}$ is connected.

Henceforth all groups of transformations satisfy (C1) - (C3).

2.1.2 Invariance under group transformations.

It is important to note that there may be real-valued functions that are unaffected by a transformation group acting on a manifold. These are known as invariants. In this section we will first give the formal definition of an invariant of a transformation group acting on a manifold. We will then present a result showing the number of functionally independent invariants, demonstrate a method for their derivation, and introduce the concept of relative and absolute invariance.

Definition 2.1.8 ([53, Definition 2.29] page 44) *Let G be a transformation group acting on a manifold M. An **invariant** of G is a real-valued function $I : M \to \mathbb{R}$ satisfying $I(\Pi(g, s)) = I(s)$ for all transformations $g \in G$ and all $s \in M$ (where Π defines the transformation group in Definition 2.1.2).*

Example 2.1.9 *If we consider Example 2.1.3 where $\Pi(g_\lambda, (x, y)) = (\lambda x, \lambda^\beta y)$, $\lambda > 0$, then the function $A_1(x, y) = \dfrac{y}{x^\beta}$ is invariant on $M = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Also $A_2(x, y) = \dfrac{x^\beta}{y}$ is invariant on $M = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and on $M = \{(x, y) \in \mathbb{R}^2 : x > 0, y < 0\}$.*

We also introduce the idea of local and global invariants.

Definition 2.1.10 ([52]) *Let G be a Lie group acting on a manifold M. A function $I : U \to \mathbb{R}$ where U an open subset of M is said to be a **local invariant** of G if $I(\Pi(g, s)) = I(s)$ for all $s \in U$ and all transformations $g \in V_s$ in some neighbourhood $V_s \subset G$ of the identity element. If $I(\Pi(g, s)) = I(s)$ for all $s \in U$ and all $g \in G$ such that $\Pi(g, s) \in U$, then I is said to be a **global invariant** of G (even though it is only defined on an open subset of M).*

Next, we give a result showing the number of functionally independent group invariants. In order to do this we require the following definition of the orbit of a transformation group.

Definition 2.1.11 *An **orbit** of a transformation group is a minimal non-empty invariant subset of M.*

(See [52] page 22 or [53] page 40.)
For a group of transformations G acting on M, $\mathcal{O} \subset M$ is an orbit if it satisfies the following:

(1) If $x \in \mathcal{O}$, and $g \in G$ then $\Pi(g, x) \in \mathcal{O}$.

(2) If $\tilde{\mathcal{O}} \subset \mathcal{O}$ and $\tilde{\mathcal{O}}$ satisfies (1), then either $\tilde{\mathcal{O}} = \mathcal{O}$ or $\tilde{\mathcal{O}}$ is empty.

In general, the orbits of a Lie group of transformations are all sub-manifolds of M. We also require the following definition.

Definition 2.1.12 ([52, Definition 1.26] page 23) *Let G be a local group of transformations acting on M.*

*(1) G acts **semi-regularly** if all its orbits have the same dimension.*

*(2) G acts **regularly** if it acts semi-regularly and also for each point $x \in M$ there exist arbitrarily small neighbourhoods whose intersection with each orbit is a connected subset thereof.*

(See, e.g., [52] pp 23-24 for examples.)
With the above definitions we now present the main result which characterises the maximum number of functionally independent invariants.

Theorem 2.1.13 ([52, Theorem 2.34] page 46) *Let G be a Lie group acting semi-regularly on the m-dimensional manifold $M(\subset \mathbb{R}^m)$ with r-dimensional orbits. At each $x \in M$ there exist $(m-r)$ functionally independent local invariants $\xi_1, \dots, \xi_{m-r}$ defined on a neighbourhood V of x. Further, any other local invariant, ξ, of the group action defined on V can be written as a function of the independent local invariants, that is $\xi = (\xi_1, \dots, \xi_{m-r})$. If G acts regularly, then the invariants can be taken to be global invariants on V.*

Example 2.1.14 *As Olver points out in [52], the orbits of the scaling transformation group acting*

$$(x_1, x_2, x_3, \dots, x_m) \to (\lambda^{\alpha_1} x_1, \lambda^{\alpha_2} x_2, \lambda^{\alpha_3} x_3, \dots, \lambda^{\alpha_m} x_m)$$

acting on the manifold $M = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^m : x_1 > 0\}$ (where $\lambda \in \mathbb{R}^+$, $\alpha_1 \neq 0$, $\alpha_2, \dots, \alpha_m \in \mathbb{R}$ not all zero, $(x_1, \dots, x_m) \in \mathbb{R}^m$) are all one-dimensional regular sub-manifolds of $\mathbb{R}^m$, except for the singular orbit consisting of $\{\mathbf{0}\}$ only. Also, the scaling group acts regularly on $\mathbb{R}^m \setminus \{\mathbf{0}\}$. Hence, there are $(m-1)$ functionally independent invariants of this group action. For the case $\alpha_1 = 1$ these are given by

$$\frac{x_2}{(x_1)^{\alpha_2}}, \frac{x_3}{(x_1)^{\alpha_3}}, \dots, \frac{x_m}{(x_1)^{\alpha_m}}.$$

Hence for Example 2.1.6 we have two functionally independent invariants, which are expressible as $\frac{y}{x^\beta}$ and $\frac{z}{x^{\beta-1}}$.

We next introduce the notion of a vector field. From this we will introduce the concept of an infinitesimal generator for a one-parameter group of transformations. It is in using the infinitesimal generator that we can determine the invariants of a given group action. In the following, $TM|_s$ is a section of the tangent bundle of M at s, a tangent bundle being the collection of all tangent spaces.

Definition 2.1.15 *A **vector field** $\boldsymbol{v}$ assigns a tangent vector $\boldsymbol{v}|_s \in TM|_s$ to each point $s \in M$, with $\boldsymbol{v}|_s$ varying smoothly from point to point.*

(See [52] page 26.)
In local coordinates $s = (s_1, \dots, s_m)$ a vector field $\boldsymbol{v}$ takes the form

$$\boldsymbol{v} = \phi_1(s)\frac{\partial}{\partial s_1} + \phi_2(s)\frac{\partial}{\partial s_2} + \cdots + \phi_m(s)\frac{\partial}{\partial s_m}, \tag{2.1.1}$$

where $\phi_1(s), \dots, \phi_m(s)$ are smooth functions of s.
An integral curve of the vector field $\boldsymbol{v}$ is a smooth parametrised curve $\sigma(\varepsilon, s) = \tilde{s}$ whose tangent vector at any point coincides with the value of $\boldsymbol{v}$ at the same point, and so

$$\frac{\partial \sigma}{\partial \varepsilon}(\varepsilon, s) = \boldsymbol{v}|_{\sigma(\varepsilon,s)}$$

for all ε, where $\boldsymbol{v}|_{\sigma(\varepsilon,s)}$ is the vector field on the curve $\sigma(\varepsilon, s)$. In local coordinates $\tilde{s} = \sigma(\varepsilon, s) = (\sigma_1(\varepsilon, s), \dots, \sigma_m(\varepsilon, s))$ must be a solution to the autonomous system of ordinary differential equations

$$\frac{\partial \tilde{s}}{\partial \varepsilon} = \phi_i(s), \qquad i = 1, \dots, m$$

where the $\phi_i(s)$ are the coefficients of $\boldsymbol{v}$ at s. From this there exists a unique maximal integral curve $\sigma : I \times M \to M$ passing through a given point $\tilde{s}_0 = \sigma(s_0, 0) \in M$. We denote the **parametrised maximal integral curve** passing through s in M by $\Upsilon(\varepsilon, s)$ and we call Υ the **flow** generated by $\boldsymbol{v}$. Thus for each $s \in M$ and ε in some interval I_s (with $0 \in I_s$) Υ is a point on the integral curve passing through s in M. Υ satisfies the following basic properties:

(F1) $\Upsilon(\varepsilon_1, \Upsilon(\varepsilon_2, s)) = \Upsilon(\varepsilon_1 + \varepsilon_2, s)$ for $s \in M$ and all ε_1, $\varepsilon_2 \in \mathbb{R}$ such that both sides of the equation makes sense;

(F2) $\Upsilon(0, s) = s$;

(F3) $\dfrac{\partial}{\partial \varepsilon}\Upsilon(\varepsilon, s) = \boldsymbol{v}|_{\Upsilon(\varepsilon,s)}$.

If we compare (F1) and (F2) with (L1) and (L2) we see that the flow generated by a vector field defines a local group action on the Lie group $\mathbb{R}$ on M (with the group operation being addition), and the vector field $\boldsymbol{v}$ for this is referred to as the **infinitesimal generator** of the group action (for more details, see [52] pp 24-31). In local coordinates,

$$\Upsilon(\varepsilon, s) = s + \varepsilon\phi(s) + O(\varepsilon^2)$$

where $\phi(s) = (\phi_1(s), \ldots, \phi_m(s))$ are the coefficients of $\boldsymbol{v}$. Thus in local coordinates the infinitesimal generator is

$$\boldsymbol{v} = \frac{\partial \sigma_1}{\partial \varepsilon}(s, \varepsilon)\frac{\partial}{\partial s_1} + \frac{\partial \sigma_2}{\partial \varepsilon}(s, \varepsilon)\frac{\partial}{\partial s_2} + \cdots + \frac{\partial \sigma_m}{\partial \varepsilon}(s, \varepsilon)\frac{\partial}{\partial s_m}. \tag{2.1.2}$$

We will demonstrate this by means of an example.

Example 2.1.16 *Consider the following one-parameter group of transformations*

$$\Pi(\varepsilon, (x, y, z, w)) = (\Xi(x, y, z, w; \varepsilon), \Phi(x, y, z, w; \varepsilon), \Gamma(x, y, z, w; \varepsilon), \Lambda(x, y, z, w; \varepsilon)) =: (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}).$$

The infinitesimal generator is

$$\begin{aligned} \boldsymbol{v} &= \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial x} + \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial y} + \left.\frac{\partial \tilde{z}}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial z} + \left.\frac{\partial \tilde{w}}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial w} \\ &= \left.\frac{\partial \Xi}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial x} + \left.\frac{\partial \Phi}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial y} + \left.\frac{\partial \Gamma}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial z} + \left.\frac{\partial \Lambda}{\partial \varepsilon}\right|_{\varepsilon=0}\frac{\partial}{\partial w}. \end{aligned}$$

We can now determine what the independent invariants are, by means of the following result.

Proposition 2.1.17 ([52, Proposition 2.6] page 81) *Let G be a connected one-parameter group of transformations acting on a manifold M, and let $f : M \to \mathbb{R}$ be an analytic real-valued function. Then f is an invariant of G if and only if $\boldsymbol{v}(f) = 0$ on M for every infinitesimal generator $\boldsymbol{v}$ of G.*

Example 2.1.18 *Consider the following scaling transformation*

$$\Pi(\varepsilon, (x, y, z, w)) = (e^{\varepsilon}x, e^{\beta\varepsilon}y, e^{(\beta-1)\varepsilon}z, e^{(\beta-2)\varepsilon}w) \tag{2.1.3}$$

on the manifold $M = \{(x, y, z, w) : x, y, z, w > 0\}$. The infinitesimal generator for this transformation is

$$\boldsymbol{v} = x\frac{\partial}{\partial x} + \beta y\frac{\partial}{\partial y} + (\beta - 1)z\frac{\partial}{\partial z} + (\beta - 2)w\frac{\partial}{\partial w}, \tag{2.1.4}$$

and so the invariants of the scaling transformation (2.1.3), Ψ_i say, must satisfy

$$\boldsymbol{v}(\Psi_i) = 0 \tag{2.1.5}$$

by Proposition 2.1.17. Now we are considering a transformation on a four-dimensional manifold M (with points represented as $(x, y, z, w) \in M \subset \mathbb{R}^4$) and the orbits of a scaling transformation are one-dimensional. Thus by Theorem 2.1.13 we are looking for three independent invariants of (2.1.3), that is three solutions of (2.1.5). It can be easily verified that

$$\Psi_1(x, y, z, w) = \frac{y}{x^{\beta}}, \ \Psi_2(x, y, z, w) = \frac{z}{x^{\beta-1}} \ \text{and} \ \Psi_3(x, y, z, w) = \frac{w}{x^{\beta-2}}$$

are functionally independent and satisfy (2.1.5), where $\boldsymbol{v}$ is given by (2.1.4). Hence $\frac{y}{x^{\beta}}$, $\frac{z}{x^{\beta-1}}$ and $\frac{w}{x^{\beta-2}}$ are three functionally independent group invariants for the scaling transformation (2.1.3).

We now consider relative and absolute invariants of a given group of transformations. We will give the definitions of a relative and an absolute invariant and state a couple of results regarding them. In order to do this we require the following definition.

Definition 2.1.19 ([53, Definition 3.12] page 82) *A function $\mu : G \times M \to \mathbb{R}$ is a* ***multiplier*** *for a transformation group G acting on a manifold M if and* *only if μ satisfies $\mu(gh, x) = \mu(g, \Pi(h, x))\mu(h, x)$ and $\mu(e, x) = 1$ for all g, $h \in G$, $x \in M$. (Recall that e is the identity element.)*

Definition 2.1.20 ([53, Definition 3.30] page 91) *Let $\mu : G \times M \to \mathbb{R}$ be a multiplier for a transformation group G acting on a manifold M. A* ***relative invariant with multiplier*** *μ is a function $R(x)$ satisfying $R(\Pi(g, x)) = \mu(g, x)R(x)$ for $g \in G$, $x \in M$.*

An invariant of a group action, defined in Definition 2.1.8, is sometimes referred to as an **absolute invariant** (see, e.g., [53] page 91). We now give two results concerning relative and absolute invariants.

Lemma 2.1.21 *If K_1 and $K_2 \neq 0$ are relative invariants with multiplier μ of a transformation group G acting on a manifold M then*

$$K = \frac{K_1}{K_2}$$

is an absolute invariant of G.

Proof: Let K_1 and K_2 be relative invariants with multiplier μ of g. Then we can say that $K_1(\Pi(g, x)) = \mu(g, x)K_1(x)$ and $K_2(\Pi(g, x)) = \mu(g, x)K_2(x)$ for any $g \in G$ and any $x \in M$, and so

$$K(\Pi(g, x)) = \frac{K_1(\Pi(g, x))}{K_2(\Pi(g, x))} = \frac{\mu(g, x)K_1(x)}{\mu(g, x)K_2(x)} = K(x).$$

Hence K is an absolute invariant of g. $\square$

Lemma 2.1.22 *Any absolute invariant K of a group transformation acting on a manifold M can be written as a function $\tilde{K}$ of the group invariants of that transformation, that is $K = \tilde{K}$(group invariants).*

This is a consequence of Theorem 2.1.13. These results will be used in §2.2 when we consider the behaviour of the integral $\mathcal{I}(\eta)$ given by (2.0.1) under a group of scaling transformations.

2.1.3 Symmetries and invariance of ordinary differential equations.

Since the Euler-Lagrange equation is a second order differential equation, we now consider second order ordinary differential equations of the form

$$\frac{d^2\eta}{dx^2}(x) = f\left(x, \eta(x), \frac{d\eta}{dx}(x)\right), \tag{2.1.6}$$

where η is a function of x, with a view to determining the most general second order ordinary differential equation that is invariant under a given one-parameter group of transformations of the form

$$\Pi(\varepsilon, (x, \eta)) = (\tilde{x}, \tilde{\eta}) = (\Xi(x, \eta; \varepsilon), \Phi(x, \eta; \varepsilon)). \tag{2.1.7}$$

In this section, given a group of transformations acting on a manifold, we will define the notion of an invariant ordinary differential equation for a given group of transformations, state a criterion for determining the invariance or otherwise of a differential equation and present a canonical form for invariant differential equations. The results and observations presented can be extended to nth order ordinary differential equations. For full details, see, e.g., [16, §3].

In order to consider second order differential equations we need to consider how the transformation group defined by (2.1.7) extends to act on derivatives of η in a natural way. This is contained in the following result.

Theorem 2.1.23 ([16, Theorem 2.3.1-1 and Theorem 2.3.1-2] page 56) *The group of transformations defined by (2.1.7) acting on a manifold M extends to its **first prolongation**, which is the one-parameter group of transformations*

$$\Pi(\varepsilon, (x, \eta, \eta')) = (\tilde{x}, \tilde{\eta}, \tilde{\eta}'),$$

where $\tilde{x}$ and $\tilde{\eta}$ are given by (2.1.7) and

$$\tilde{\eta}' = \Phi_1(x, \eta, \eta'; \varepsilon) = \frac{\dfrac{\partial \Phi}{\partial x} + \eta' \dfrac{\partial \Phi}{\partial \eta}}{\dfrac{\partial \Xi}{\partial x} + \eta' \dfrac{\partial \Xi}{\partial \eta}}. \tag{2.1.8}$$

*The group of transformations defined by (2.1.7) extends further to its **second prolongation**, which is the one-parameter group of transformations*

$$\Pi(\varepsilon, (x, \eta, \eta', \eta'')) = (\tilde{x}, \tilde{\eta}, \tilde{\eta}', \tilde{\eta}''),$$

where $\tilde{x}$ and $\tilde{\eta}$ are given by (2.1.7), $\tilde{\eta}'$ is given by (2.1.8) and

$$\tilde{\eta}'' = \Phi_2(x, \eta, \eta', \eta''; \varepsilon) = \frac{\dfrac{\partial \Phi_1}{\partial x} + \eta' \dfrac{\partial \Phi_1}{\partial \eta} + \eta'' \dfrac{\partial \Phi_1}{\partial \eta'}}{\dfrac{\partial \Xi}{\partial x} + \eta' \dfrac{\partial \Xi}{\partial \eta}}. \tag{2.1.9}$$

Example 2.1.24 *If we consider the scaling group* $\Pi(\varepsilon,(x,\eta)) = (e^{\varepsilon}x, e^{\beta\varepsilon}\eta)$ *then*

$$\tilde{\eta}' = \Phi_1 = \frac{\eta' e^{\beta\varepsilon}}{e^{\varepsilon}} = e^{(\beta-1)\varepsilon}\eta' \quad and \quad \tilde{\eta}'' = \Phi_2 = \frac{\eta''\dfrac{\partial\Phi_1}{\partial\eta'}}{\dfrac{\partial\Xi}{\partial x}} = \frac{\eta'' e^{(\beta-1)\varepsilon}}{e^{\varepsilon}} = e^{(\beta-2)\varepsilon}\eta''.$$

Hence, the first prolongation of the scaling transformation is given by

$$\Pi(\varepsilon,(x,\eta,\eta')) = (e^{\varepsilon}x, e^{\beta\varepsilon}\eta, e^{(\beta-1)\varepsilon}\eta'), \tag{2.1.10}$$

and the second prolongation of the scaling transformation is given by

$$\Pi(\varepsilon,(x,\eta,\eta',\eta'')) = (e^{\varepsilon}x, e^{\beta\varepsilon}\eta, e^{(\beta-1)\varepsilon}\eta', e^{(\beta-2)\varepsilon}\eta''). \tag{2.1.11}$$

We now remark on the notation used in presenting differential equations both in this section and in §2.2.2.

Remark 2.1.25 *If in our expressions for differential equations we use* $\eta(x)$, $\dfrac{d\eta}{dx}(x)$ *and* $\dfrac{d^2\eta}{dx^2}(x)$ *then in that expression we are considering* η *as a function of* x*, and thus we have one independent variable. If instead we use* η*,* η' *and* η'' *then we are considering* x*,* η*,* η' *and* η'' *as independent variables, related by an algebraic expression. Thus, if we write the second order differential equation as*

$$\frac{d^2\eta}{dx^2}(x) = f\left(x, \eta(x), \frac{d\eta}{dx}(x)\right) \tag{2.1.12}$$

then we treat (2.1.12) as a differential equation, with η *being a function of* x*, and if we write*

$$\eta'' = f(x,\eta,\eta') \tag{2.1.13}$$

then we treat (2.1.13) as an algebraic equation, with η'' *being a function of* $(x,\eta,\eta') \in \mathbb{R}^3$.

We next define the notion of an invariant second order ordinary differential equation.

Definition 2.1.26 ([16, Definition 3.1.1-1] page 90) *The group of transformations defined by (2.1.7) leaves the second order ordinary differential equation (2.1.6) invariant if and only if its second prolongation leaves the surface* $\eta'' = f(x,\eta,\eta')$ *invariant.*

Remark 2.1.27 *One consequence of a differential equation being invariant under a transformation group* $\Pi(\varepsilon, (x, \eta))$ *acting on* $M \subset \mathbb{R}^2$ *is that* $\tilde{\eta}(\tilde{x})$ *is a solution of the differential equation whenever* $\eta(x)$ *is (see, e.g., [16]).*

The next theorem is the infinitesimal criterion for invariance of a second order ordinary differential equation. It provides a way of determining whether or not an ordinary differential equation is invariant under a given group of transformations acting on a manifold by making use of the infinitesimal generator for the prolongations of the group of transformations. In order to do this we consider the one-parameter group of transformations where

$$\Xi(x, \eta; \varepsilon) = x + \varepsilon\xi(x, \eta) + O(\varepsilon^2) \text{ and } \Phi(x, \eta; \varepsilon) = \eta + \varepsilon\varphi(x, \eta) + O(\varepsilon^2). \tag{2.1.14}$$

This group of transformations has as its infinitesimal generator

$$\boldsymbol{v} = \xi(x, \eta)\frac{\partial}{\partial x} + \varphi(x, \eta)\frac{\partial}{\partial \eta}. \tag{2.1.15}$$

If we write Φ_1 and Φ_2 as

$$\begin{aligned} \Phi_1(x, \eta, \eta'; \varepsilon) &= \eta' + \varepsilon\varphi_1(x, \eta, \eta') + O(\varepsilon^2), \\ \Phi_2(x, \eta, \eta', \eta''; \varepsilon) &= \eta'' + \varepsilon\varphi_2(x, \eta, \eta', \eta'') + O(\varepsilon^2), \end{aligned} \tag{2.1.16}$$

then the first prolonged infinitesimal generator is

$$\boldsymbol{v}' = \xi(x, \eta)\frac{\partial}{\partial x} + \varphi(x, \eta)\frac{\partial}{\partial \eta} + \varphi_1(x, \eta, \eta')\frac{\partial}{\partial \eta'}, \tag{2.1.17}$$

and the second prolonged infinitesimal generator is

$$\boldsymbol{v}'' = \xi(x, \eta)\frac{\partial}{\partial x} + \varphi(x, \eta)\frac{\partial}{\partial \eta} + \varphi_1(x, \eta, \eta')\frac{\partial}{\partial \eta'} + \varphi_2(x, \eta, \eta', \eta'')\frac{\partial}{\partial \eta''}. \tag{2.1.18}$$

We now make use of the following (for the general result see [16, Theorem 2.3.2-1]):

Theorem 2.1.28 *Let* $\Pi(\varepsilon, (x, \eta)) = (\Xi(x, \eta; \varepsilon), \Phi(x, \eta; \varepsilon))$ *be of the form (2.1.14) and let* $\phi_1(x, \eta, \eta'; \varepsilon)$ *and* $\phi_2(x, \eta, \eta', \eta''; \varepsilon)$ *be given by (2.1.16). Then*

$$\varphi_1(x, \eta, \eta') = \frac{\partial \varphi}{\partial x}(x, \eta) + \eta'\frac{\partial \varphi}{\partial \eta}(x, \eta) - \eta'\frac{\partial \xi}{\partial x}(x, \eta) - (\eta')^2\frac{\partial \xi}{\partial \eta}(x, \eta)$$

and

$$\varphi_2(x,\eta,\eta',\eta'') = \frac{\partial \varphi_1}{\partial x}(x,\eta,\eta') + \eta'\frac{\partial \varphi_1}{\partial \eta}(x,\eta,\eta')+\eta''\frac{\partial \varphi_1}{\partial \eta'}(x,\eta,\eta') \\ -\eta''\frac{\partial \xi}{\partial x}(x,\eta) - \eta'\eta''\frac{\partial \xi}{\partial \eta}(x,\eta).$$

Upon applying Theorem 2.1.28, explicit forms for φ_1 and φ_2 are

$$\begin{aligned} \varphi_1 &= \varphi_{,x}+(\varphi_{,\eta} - \xi_{,x})\eta' - \xi_{,\eta}(\eta')^2 \\ \varphi_2 &= \varphi_{,xx}+(2\varphi_{,x\eta} - \xi_{,xx})\eta' + (\varphi_{,\eta\eta} - 2\xi_{,x\eta})(\eta')^2 \\ &\qquad - \xi_{,\eta\eta}(\eta')^3(\varphi_{,\eta} - 2\xi_{,x})\eta'' - 3\xi_{,\eta}\eta'\eta''. \end{aligned} \tag{2.1.19}$$

We are now in a position to state the infinitesimal criterion for invariance of a second order ordinary differential equation. (For the case of an nth order differential equation see, e.g., [16, Theorem 3.1.1-1] page 91.)

Theorem 2.1.29 *Let $\boldsymbol{v}$ as given by (2.1.15) be the infinitesimal generator of (2.1.7), and let $\boldsymbol{v}''$ as given by (2.1.18) be the second prolonged infinitesimal generator of (2.1.15), where $\varphi_1(x,\eta,\eta')$ and $\varphi_2(x,\eta,\eta',\eta'')$ are given by (2.1.19). Then the transformation group acting on M given by (2.1.7) leaves the ordinary differential equation (2.1.6) invariant if and only if*

$$\boldsymbol{v}''\left(\eta'' - f\left(x,\eta,\eta'\right)\right) = 0$$

where $f : \mathbb{R}^3 \to \mathbb{R}$.

We now consider the problem of finding the most general second order ordinary differential equation which is invariant under a given one-parameter group of transformations. If

$$\frac{d^2\eta}{dx^2}(x) = f\left(x, \eta(x), \frac{d\eta}{dx}(x)\right)$$

is invariant under the group of transformations $G := \{\Pi(\varepsilon,(x,\eta)) : \varepsilon \in \mathbb{R}\}$, where $\Pi(\varepsilon,(x,\eta)) = (\tilde{x},\tilde{\eta})$, then $\boldsymbol{v}''(\eta'' - f(x,\eta,\eta')) = 0$ by Theorem 2.1.29. Moreover, by Theorem 2.1.13 and Proposition 2.1.17, if $\boldsymbol{v}''(F(x,\eta,\eta',\eta'')) = 0$ then $F(x,\eta,\eta',\eta'') = \tilde{F}(\text{group invariants of } G'')$ where G'' is the group of second prolongations of G. If we let p, q and r be three independent group invariants of G'', and put $F(x,\eta,\eta',\eta'') = \eta'' - f\left(x,\eta,\eta'\right)$ then $\eta'' - f\left(x,\eta,\eta'\right) = 0$ implies that

$$\tilde{F}(p, q, r) = 0, \tag{2.1.20}$$

and, since the differential equation is of the form (2.1.6), we can rewrite (2.1.20) as

$$p = \hat{F}(q, r).$$

We will demonstrate this by means of an example.

Example 2.1.30 *Suppose the second order ordinary differential equation*

$$\frac{d^2\eta}{dx^2}(x) - f\left(x, \eta(x), \frac{d\eta}{dx}(x)\right) = 0 \tag{2.1.21}$$

is invariant under the scaling group given in Example 2.1.24. Then, putting $F(x, \eta, \eta', \eta'') = \eta'' - f(x, \eta, \eta')$, *we have that*

$$\boldsymbol{v}''(F) = x\frac{\partial F}{\partial x} + \beta\eta\frac{\partial F}{\partial \eta} + (\beta - 1)\eta'\frac{\partial F}{\partial \eta'} + (\beta - 2)\eta''\frac{\partial F}{\partial \eta''} = 0. \tag{2.1.22}$$

Now, by Example 2.1.18 three functionally independent invariants of the second prolongation (given by (2.1.11)) are $\frac{\eta}{x^\beta}$, $\frac{\eta'}{x^{\beta-1}}$ *and* $\frac{\eta''}{x^{\beta-2}}$. *Thus the second order equation (2.1.21) can be rewritten as*

$$\tilde{F}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}, \frac{\eta''}{x^{\beta-2}}\right) = 0$$

or

$$\frac{\eta''}{x^{\beta-2}} = \hat{F}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right).$$

2.1.4 Change of coordinates.

Suppose we make a change of coordinates from $s = (s_1, \ldots, s_m)$ to $(t_1, \ldots, t_m) = t$ where $t_i = \nu_i(s)$. Then the infinitesimal generators in the original and new coordinates are related as follows.

Remark 2.1.31 *The infinitesimal generator for a one-parameter transformation*

$$\Pi(\varepsilon, (s_1, \ldots, s_m)) = (\tilde{s}_1, \ldots, \tilde{s}_m)$$

acting on $M \subset \mathbb{R}^m$ in the local coordinate system $s = (s_1, \dots, s_m)$ is

$$\boldsymbol{v} = \sum_{i=1}^{m} \left(\left. \frac{\partial \tilde{s}_i}{\partial \varepsilon} \right|_{\varepsilon=0} \right) \frac{\partial}{\partial s_i}.$$

If $t = (t_1, \dots, t_m) = (\nu_1(s), \dots, \nu_m(s)) = \nu(s)$ then the infinitesimal generator for the transformation in terms of the new coordinates is given by

$$\tilde{\boldsymbol{v}} = \sum_{j=1}^{m} \left(\sum_{i=1}^{m} \left(\left. \frac{\partial \tilde{s}_i}{\partial \varepsilon} \right|_{\varepsilon=0} \right) \frac{\partial t_j}{\partial s_i} \right) \frac{\partial}{\partial t_j} = \sum_{j=1}^{m} \boldsymbol{v}(t_j) \frac{\partial}{\partial t_j}.$$

We now present a key result regarding a change of coordinates which has proved useful in integrating ordinary differential equations using Lie group theory. We will use this in our study of classical variational problems (see §2.2.3).

Proposition 2.1.32 ([52, Proposition 1.29] page 30) *Let M be a smooth m-dimensional manifold in $\mathbb{R}^m$. Suppose that a vector field $\boldsymbol{v}$ does not vanish at a point $x_0 \in M$. Then there is a local coordinate chart $y = (y_1, \dots, y_m)$ at x_0 such that in terms of these coordinates the vector field is*

$$\boldsymbol{v} = \frac{\partial}{\partial y_1}.$$

(See also, e.g., [16, Theorem 2.2.5-3] page 45 or [40] page 95.)
Thus it is possible to make a change of variables from $x = (x_1, \dots, x_m)$ to $y = (y_1, \dots, y_m)$ such that any one-parameter group of transformations is a one-parameter group of translations in terms of these new coordinates, which can be written as

$$\tilde{y}_1 = y_1 + \nu, \qquad \tilde{y}_i = y_i, \ \ i = 2, \dots, m,$$

and thus

$$\Pi(\nu, (y_1, y_2, \dots, y_m)) = (y_1 + \nu, y_2, \dots, y_m).$$

We will demonstrate this with an example.

Example 2.1.33 *Consider the set of transformations $G = \{\Pi(\varepsilon, (x, \eta, \chi)) : \varepsilon \in \mathbb{R}\}$, where $\Pi(\varepsilon, (x, \eta, \chi)) = (e^{\varepsilon} x, e^{\beta\varepsilon} \eta, e^{(\beta-1)\varepsilon} \chi)$ for each $\varepsilon \in \mathbb{R}$. By Proposition 2.1.32 we can choose a change of variables (y, w, z) with $y = \rho(x, \eta, \chi)$, $w =$*

$\sigma(x,\eta,\chi)$ and $z=\tau(x,\eta,\chi)$ such that the infinitesimal generator is of the form

$$\hat{\boldsymbol{v}} = \frac{\partial}{\partial w}, \tag{2.1.23}$$

and so the set of transformations in this new coordinate system is

$$\hat{G} = \{\Pi(\nu,(y,w,z)) : \nu \in \mathbb{R}\}$$

(ν is the group parameter of $\hat{G}$), where for each $\nu \in \mathbb{R}$,

$$\Pi(\nu,(y,w,z)) = (y, w+\nu, z) = (\hat{y},\hat{w},\hat{z}). \tag{2.1.24}$$

By Remark 2.1.31, we can say that

$$\hat{\boldsymbol{v}} = \tilde{\boldsymbol{v}}(y)\frac{\partial}{\partial y} + \tilde{\boldsymbol{v}}(w)\frac{\partial}{\partial w} + \tilde{\boldsymbol{v}}(z)\frac{\partial}{\partial z}.$$

Now, in order to obtain (2.1.23) we require $\tilde{\boldsymbol{v}}(y)=0$, $\tilde{\boldsymbol{v}}(w)=1$ and $\tilde{\boldsymbol{v}}(z)=0$. As $\tilde{\boldsymbol{v}}$ is of the form

$$\tilde{\boldsymbol{v}} = x\frac{\partial}{\partial x} + \beta\eta\frac{\partial}{\partial \eta} + (\beta-1)\chi\frac{\partial}{\partial \chi}, \tag{2.1.25}$$

these three equations are satisfied by putting

$$y = \frac{\eta}{x^\beta},\ w = \log x \text{ and } z = \frac{x^\beta}{x\chi - \beta\eta}. \tag{2.1.26}$$

Note that z is a function of $\frac{\eta}{x^\beta}$ and $\frac{\chi}{x^{\beta-1}}$.

2.2 Application to one-dimensional variational problems.

2.2.1 Relative invariance of integrals under a scaling transformation.

We now return to consider integrals of the form

$$\mathcal{I}(\eta) = \int_0^b L(x,\eta(x),\eta'(x))dx. \tag{2.2.1}$$

In (2.2.1) η and η' are functions of x. If we consider the transformation

$$\Pi(\varepsilon, (x, \eta)) = (\Xi(x, \eta; \varepsilon), \Phi(x, \eta; \varepsilon)) =: (\tilde{x}, \tilde{\eta})$$

then as η is a function of x this transformation induces an action $\eta' \to \tilde{\eta}'$ (see §2.1.3). Thus, if we replace x, $\eta(x)$ & $\eta'(x)$ by $\tilde{x}$, $\tilde{\eta}(\tilde{x})$ & $\tilde{\eta}'(\tilde{x})$, as obtained by this transformation, then as x is defined on $[0, b]$, $\tilde{x}$ is defined on a corresponding interval $[\tilde{a}, \tilde{b}]$, and the integral $\mathcal{I}(\eta)$ given by (2.2.1) becomes $\tilde{\mathcal{I}}(\tilde{\eta})$ where

$$\tilde{\mathcal{I}}(\tilde{\eta}) = \int_{\tilde{a}}^{\tilde{b}} L(\tilde{x}, \tilde{\eta}(\tilde{x}), \tilde{\eta}'(\tilde{x}))d\tilde{x}. \tag{2.2.2}$$

We recall that in §2.1.3 we found the most general form for an ordinary differential equation invariant under a given group of transformations. We also remark that in the examples of [60] and [68] the integral $\mathcal{I}(\eta)$ can be written in the form

$$\mathcal{I}(\eta) = \int_0^b L(x, \eta(x), \eta'(x))\, dx = \int_0^b x^p \tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)\, dx \tag{2.2.3}$$

and that $\dfrac{\eta}{x^\beta}$ and $\dfrac{\eta'}{x^{\beta-1}}$ are invariants under the scaling transformation

$$\Pi(\varepsilon, (x, \eta, \eta')) = (e^\varepsilon x, e^{\beta\varepsilon}\eta, e^{(\beta-1)\varepsilon}\eta').$$

With this in mind we now determine the conditions on L such that $\mathcal{I}(\eta)$ can be rewritten in the form (2.2.3). In order to do this we consider $\Pi(\varepsilon, (x, \eta))$ of the form

$$\Pi(\varepsilon, (x, \eta)) = (e^\varepsilon x, e^{\beta\varepsilon}\eta) = (\lambda x, \lambda^\beta \eta) =: \hat{\Pi}(\lambda, (x, \eta)), \qquad \lambda = e^\varepsilon > 0. \tag{2.2.4}$$

Henceforth, we will define $\hat{\Pi}(\lambda, (x, \eta))$ by (2.2.4) and H to be the set of scaling transformations $H := \{\hat{\Pi}(\lambda, (x, \eta)) : \lambda \in \mathbb{R}^+\}$. Note that as $\lambda = e^\varepsilon$ the identity transformation occurs when $\lambda = 1$. Since the integrand in (2.2.1) is a function of x, η & η', we need to look at the first prolongation of the group transformation. From Example 2.1.24, the first prolongation of the scaling transformation $\Pi(\varepsilon, (x, \eta))$ (where λ is fixed) is

$$\Pi(\varepsilon, (x, \eta, \eta')) = (\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta') =: \hat{\Pi}(\lambda, (x, \eta, \eta')), \qquad \lambda = e^\varepsilon > 0, \tag{2.2.5}$$

and we define $H' := \{\hat{\Pi}(\lambda, (x, \eta, \eta')) : \lambda \in \mathbb{R}^+\}$. Thus

$$\tilde{\mathcal{I}}(\tilde{\eta}) = \int_0^{\tilde{b}} L(\tilde{x}, \tilde{\eta}(\tilde{x}), \tilde{\eta}'(\tilde{x}))d\tilde{x} = \int_0^b \lambda L(\lambda x, \lambda^\beta \eta(x), \lambda^{\beta-1}\eta'(x))dx. \tag{2.2.6}$$

Definition 2.2.1 *We will say that the integral functional*

$$\mathcal{I}(\eta) = \int_c^d L(x, \eta(x), \eta'(x))dx$$

satisfies the scaling condition $(\mathcal{S}_n)$ *if, on considering the scaling transformation (2.2.4), we have that*

$$\tilde{\mathcal{I}}(\tilde{\eta}) = \lambda^n \mathcal{I}(\eta) \tag{2.2.7}$$

for some fixed n*, for all* $\lambda > 0$*, for all* $c \geq 0$ *&* $d > 0$ *and all* η *and* η' *(where* $\tilde{\eta} = \lambda^\beta \eta$*).*

Lemma 2.2.2 $\mathcal{I}(\eta)$ *satisfies* $(\mathcal{S}_n)$ *(for* $n \in \mathbb{R}$*) if and only if* L *is a relative invariant with multiplier* $\mu = \lambda^{n-1}$ *of the set of scaling transformations* H*.*

Proof: If $\mathcal{I}(\eta)$ satisfies $(\mathcal{S}_n)$ then we can write

$$\int_c^d \lambda L(\lambda x, \lambda^\beta \eta(x), \lambda^{\beta-1}\eta'(x))dx = \lambda^n \int_c^d L(x, \eta(x), \eta'(x))dx, \tag{2.2.8}$$

and since (2.2.8) holds for all values of $c \geq 0$ and $d > 0$, we have that

$$L(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta') = \lambda^{n-1} L(x, \eta, \eta') \tag{2.2.9}$$

for some fixed n, for all $\lambda > 0$, for all $x \geq 0$ and for all η & η'. Thus L is a relative invariant with multiplier $\mu = \lambda^{n-1}$ of the scaling group H. Conversely, if L is a relative invariant with multiplier $\mu = \lambda^p$ of the scaling group H then it is easy to show that $\mathcal{I}(\eta)$ satisfies $(\mathcal{S}_n)$ with $n = p + 1$. □

All the examples (I)-(III) satisfy (2.2.9). We now present the main result in this section.

Proposition 2.2.3 *Suppose that* $L(x, \eta, \eta')$ *is a relative invariant with multiplier* λ^p *(where* p *is fixed and* $\lambda > 0$*) of the group of first prolongations* H' *of the scaling transformations* H*. Then* $L(x, \eta, \eta')$ *can be written in the form*

$$L(x, \eta, \eta') = x^p \tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right). \tag{2.2.10}$$

Proof: Fix p and let L be a relative invariant with multiplier λ^p of H'. Then L is such that $L(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta') = \lambda^p L(x, \eta, \eta')$. Now we observe that $L'(x, \eta, \eta') = x^p$ is also a relative invariant with multiplier λ^p of H' for the same value of p (as $L'(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta') = \lambda^p x^p = \lambda^p L'(x, \eta, \eta'))$. Hence by Lemma 2.1.21,

$$\frac{L(x, \eta, \eta')}{x^p}$$

is an absolute invariant of H' for all $\lambda > 0$, and hence by Lemma 2.1.22 we can write

$$\frac{L(x, \eta, \eta')}{x^p} = \tilde{L}(\text{group invariants of } H'). \tag{2.2.11}$$

We recall from Example 2.1.18 that $\frac{\eta}{x^\beta}$ and $\frac{\eta'}{x^{\beta-1}}$ are two functionally independent group invariants of H'. Hence we can rewrite (2.2.11) as

$$\frac{L(x, \eta, \eta')}{x^p} = \tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right), \tag{2.2.12}$$

and therefore we can rewrite (2.2.12) as (2.2.10). □

Corollary 2.2.4 *If $\mathcal{I}(\eta)$ satisfies $(\mathcal{S}_n)$ we can express $\mathcal{I}(\eta)$ in the form*

$$\mathcal{I}(\eta) = \int_0^b x^{n-1}\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)\, dx. \tag{2.2.13}$$

Remark 2.2.5 *We observe that for η such that* (B1) *holds and $\mathcal{I}(\eta)$ of the form (2.2.13), the integrand $\tilde{L}$ satisfies the following relation*

$$\frac{d}{dx}\left\{\frac{x^n}{n}\left[\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) + \left(\frac{\beta\eta(x)}{x^\beta} - \frac{\eta'(x)}{x^{\beta-1}}\right)\tilde{L}_{,2}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)\right]\right\} = x^{n-1}\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right), \tag{2.2.14}$$

and hence

$$\begin{aligned}\mathcal{I}(\eta) &= \int_0^b x^{n-1}\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)dx \\ &= \frac{x^n}{n}\left[\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) + \left(\frac{\beta\eta(x)}{x^\beta} - \frac{\eta'(x)}{x^{\beta-1}}\right)\tilde{L}_{,2}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)\right]\Bigg|_0^b.\end{aligned} \tag{2.2.15}$$

This is as a consequence of relative scale invariance and is related to Green's identity (cf [8], [60]).

2.2.2 Relative invariance and invariance of the Euler-Lagrange equation

We will now show that relative invariance of the integrand in (2.2.1) implies invariance of the Euler-Lagrange equation. Let $L(x, \eta, \eta')$ be a relative invariant with multiplier λ^p, p fixed, of H'. Then, by Proposition 2.2.3, L can be written as in (2.2.10). Since the Euler-Lagrange equation is a second order differential equation in η we need to consider the second prolongation of the group action. From Example 2.1.24, the second prolongation of the set of scaling transformations H is

$$\begin{aligned}\Pi(\varepsilon, (x, \eta, \eta', \eta'')) &= (e^{\varepsilon}x, e^{\beta\varepsilon}\eta, e^{(\beta-1)\varepsilon}\eta', e^{(\beta-2)\varepsilon}\eta'') \\ &= (\lambda x, \lambda^{\beta}\eta, \lambda^{\beta-1}\eta', \lambda^{\beta-2}\eta'') =: \hat{\Pi}(\lambda, (x, \eta, \eta', \eta'')),\ \lambda = e^{\varepsilon} > 0\end{aligned} \tag{2.2.16}$$

with $\lambda > 0$ fixed, and we define $H'' := \{\hat{\Pi}(\lambda, (x, \eta, \eta', \eta'')) : \lambda \in \mathbb{R}^+\}$.
Now, the Euler-Lagrange equation for $\mathcal{I}(\eta)$ given by (2.2.1) is

$$L_{,2}\left(x, \eta(x), \frac{d\eta}{dx}(x)\right) - \frac{d}{dx}\left[L_{,3}\left(x, \eta(x), \frac{d\eta}{dx}(x)\right)\right] = 0, \tag{2.2.17}$$

which can be rewritten as

$$L_{,2}(x, \eta, \eta') - L_{,31}(x, \eta, \eta') - L_{,32}(x, \eta, \eta')\eta' - L_{,33}(x, \eta, \eta')\eta'' = 0. \tag{2.2.18}$$

(See Remark 2.1.25 for an explanation of the notation.) The next result shows that $\mathcal{I}(\eta)$ satisfying $(\mathcal{S}_n)$ implies invariance of the Euler-Lagrange equation for $\mathcal{I}(\eta)$.

Proposition 2.2.6 *Let $\mathcal{I}(\eta)$ satisfy $(\mathcal{S}_n)$. Then the Euler-Lagrange equation for $\mathcal{I}(\eta)$, that is (2.2.17), is invariant under the group of scaling transfornations H.*

Proof: By Definition 2.1.26, it is enough to show that (2.2.18) is invariant under the group H''. Let $L(x, \eta, \eta')$ be a relative invariant with multiplier λ^p under the set of extended scaling transformations H'. Then, by Proposition

2.2.3, $\mathcal{I}(\eta)$ can be rewritten as

$$\mathcal{I}(\eta) = \int_0^b x^p \tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) dx. \tag{2.2.19}$$

The Euler-Lagrange equation (2.2.17) for $\mathcal{I}(\eta)$ of the form (2.2.19) is

$$\frac{\partial}{\partial \eta}\left(x^p \tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{1}{x^{\beta-1}}\frac{d\eta}{dx}(x)\right)\right) - \frac{d}{dx}\left(\frac{\partial}{\partial \eta'}\left(x^p \tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{1}{x^{\beta-1}}\frac{d\eta}{dx}(x)\right)\right)\right) = 0 \tag{2.2.20}$$

and (2.2.18) can be rewritten as

$$\begin{aligned} x^{p-\beta}\Bigg[&\tilde{L}_{,1}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) - (p+1-\beta)\tilde{L}_{,2}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) \\ &- \left\{\frac{\eta'}{x^{\beta-1}} - \frac{\beta\eta}{x^\beta}\right\}\tilde{L}_{,21}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) - \left\{\frac{\eta''}{x^{\beta-2}} - (\beta-1)\frac{\eta'}{x^{\beta-1}}\right\}\tilde{L}_{,22}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right)\Bigg] \\ &= 0. \end{aligned} \tag{2.2.21}$$

Now, from Example 2.1.30, $\frac{\eta}{x^\beta}$, $\frac{\eta'}{x^{\beta-1}}$ and $\frac{\eta''}{x^{\beta-2}}$ are all invariants of H'' and hence

$$\tilde{L}_{,1}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right), \ \tilde{L}_{,2}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right), \ \tilde{L}_{,21}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) \text{ and } \tilde{L}_{,22}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right)$$

are invariants of H'', and thus (2.2.20) is scale invariant. □

The following example shows that the converse to the above is false in general.

Example 2.2.7 *Consider the integral*

$$\bar{\mathcal{I}}(\eta) = \int_0^b \left(\eta(x) - \frac{x^2}{2} - x\frac{d\eta}{dx}(x)\right)^2 + \frac{d\eta}{dx}(x) - x \ dx.$$

The Euler-Lagrange equation $4\eta(x) + 2x^2 - 2x^2\frac{d^2\eta}{dx^2}(x) = 0$ *is invariant under the scaling transformation* $\hat{\Pi}(\lambda, (x, \eta)) = (\lambda x, \lambda^2\eta)$ *but* $\mathcal{I}(\eta)$ *does not satisfy* $(\mathcal{S}_n)$.

The next result shows that invariance of the Euler-Lagrange equation implies the existence of at least one relative invariant integrand.

Proposition 2.2.8 *Let*

$$\frac{d^2\eta}{dx^2}(x) = F\left(x, \eta(x), \frac{d\eta}{dx}(x)\right) \tag{2.2.22}$$

be a second order ordinary differential equation that is invariant under the set of scaling transformations H (as defined in §2.2.1). Then for $n > 0$ there exists an integrand $\hat{L}(x, \eta, \eta')$ such that $\hat{L}$ is a relative invariant with multiplier λ^n of H', and (2.2.22) is the Euler-Lagrange equation for

$$\hat{\mathcal{I}}(\eta) = \int_0^b \hat{L}(x, \eta(x), \eta'(x))dx. \tag{2.2.23}$$

Proof: The proof uses the approach of [17] pp 30-32 and Definition 2.1.26. The general solution $\eta(x)$ of the second order differential equation (2.2.22) is of the form

$$\eta(x) = f(x, \sigma, \tau), \tag{2.2.24}$$

where f is C^2 and σ and τ are constants of integration. As (2.2.22) is scale invariant it can be rewritten as

$$\frac{d^2\eta}{dx^2}(x) = x^{\beta-2}\Gamma\left(\frac{\eta(x)}{x^\beta}, \frac{1}{x^{\beta-1}}\frac{d\eta}{dx}(x)\right), \tag{2.2.25}$$

As (2.2.25) is scale invariant,

$$\tilde{\eta}(\tilde{x}) = \lambda^\beta \eta\left(\frac{\tilde{x}}{\lambda}\right)$$

is also a solution of (2.2.25) whenever η is given by (2.2.24). Our aim is to determine an integrand $\hat{L}(x, \eta, \eta')$ such that (2.2.25) becomes identical with the Euler-Lagrange equation for (2.2.23). Hence we require that the algebraic equation

$$\hat{L}_{,2}(x,\eta,\eta') - \hat{L}_{,31}(x,\eta,\eta') - \eta'\hat{L}_{,32}(x,\eta,\eta') = x^{\beta-2}\Gamma\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right)\hat{L}_{,33}(x,\eta,\eta') \tag{2.2.26}$$

holds for any choice of $(x, \eta, \eta') \in \mathbb{R}^3$. Differentiation of (2.2.26) with respect to η' and putting $M(x, \eta, \eta') = \hat{L}_{,33}(x, \eta, \eta')$ gives us that

$$\begin{aligned} M_{,1}(x,\eta,\eta') + \eta' M_{,2}(x,\eta,\eta') &+ x^{\beta-2}\Gamma\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) M_{,3}(x,\eta,\eta') \\ &+ x^{-1}\Gamma_{,2}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) M(x,\eta,\eta') = 0. \end{aligned} \tag{2.2.27}$$

We proceed as follows: along a solution of (2.2.22) of the form (2.2.24), it follows that (2.2.25) holds. Hence, along the solution $\eta(x) = f(x, \sigma, \tau)$ we can rewrite (2.2.27) as a differential equation of the form

$$\frac{d}{dx}\left[M\left(x, \eta(x), \frac{d\eta}{dx}(x)\right)\right] + \frac{1}{x}\Gamma_{,2}\left(\frac{\eta(x)}{x^\beta}, \frac{1}{x^{\beta-1}}\frac{d\eta}{dx}(x)\right) M\left(x, \eta(x), \frac{d\eta}{dx}(x)\right) = 0. \tag{2.2.28}$$

Now, let us write the integrating factor for (2.2.28) as

$$\begin{aligned}\theta(x, \sigma, \tau) &= \exp\left(\int^x F_{,3}(t, f(t, \sigma, \tau), f_{,1}(t, \sigma, \tau))dt\right)\\ &= \exp\left(\int^x t^{-1}\Gamma_{,2}\left(\frac{f(t, \sigma, \tau)}{t^\beta}, \frac{f_{,1}(t, \sigma, \tau)}{t^{\beta-1}}\right) dt\right).\end{aligned} \tag{2.2.29}$$

Hence

$$\theta(x, \sigma, \tau) M\left(x, \eta(x), \frac{d\eta}{dx}(x)\right) = \Psi(\sigma, \tau) \tag{2.2.30}$$

where Ψ is an arbitrary function of σ and τ. Now, by Definition 2.1.26, the second order differential equation (2.2.22) is an invariant of H if and only if the algebraic equation

$$\eta'' = F(x, \eta, \eta') \tag{2.2.31}$$

is an invariant of H''. Now, we assume that the algebraic equations $\eta = f(x, \sigma, \tau)$ and $\eta' = f_{,1}(x, \sigma, \tau)$ can be solved to obtain $\sigma = \phi(x, \eta, \eta')$ and $\tau = \psi(x, \eta, \eta')$. (This follows from the existence and uniqueness of solutions to the initial value problem.) Thus we can write

$$\theta(x, \phi(x, \eta, \eta'), \psi(x, \eta, \eta'))M(x, \eta, \eta') = \Psi(\phi(x, \eta, \eta'), \psi(x, \eta, \eta')). \tag{2.2.32}$$

Also we can write

$$\begin{aligned}&\chi(x, \eta, \eta') := \theta(x, \phi(x, \eta, \eta'), \psi(x, \eta, \eta'))\\ &= \exp\left(\int^x \frac{1}{t}\Gamma_{,2}\left(\frac{f(t, \phi(x, \eta, \eta'), \psi(x, \eta, \eta'))}{t^\beta}, \frac{f_{,1}(t, \phi(x, \eta, \eta'), \psi(x, \eta, \eta'))}{t^{\beta-1}}\right) dt\right).\end{aligned}$$

Now, from (2.2.30) we have that

$$M(x,\eta,\eta') = \frac{1}{\chi(x,\eta,\eta')}\Psi(\phi(x,\eta,\eta'),\psi(x,\eta,\eta')), \tag{2.2.33}$$

with Ψ being an arbitrary function in ϕ and ψ. Now, replacing (x,η,η') by $(\tilde{x},\tilde{\eta},\tilde{\eta}')$ means we can rewrite (2.2.33) as

$$M(\tilde{x},\tilde{\eta},\tilde{\eta}') = \frac{1}{\chi(\tilde{x},\tilde{\eta},\tilde{\eta}')}\Psi(\phi(\tilde{x},\tilde{\eta},\tilde{\eta}'),\psi(\tilde{x},\tilde{\eta},\tilde{\eta}')). \tag{2.2.34}$$

As σ and τ are constants of integration, and as η is defined on $[0,b]$, we will assume without loss of generality that $\sigma = \eta(0)$ and $\tau = \eta'(0)$. (Note that this assumes that a solution to the initial value problem exists for all x in the interval under consideration.) Now, as a solution of (2.2.25) is of the form (2.2.24), we can write $\tilde{\eta}(\tilde{x})$ in the form $\tilde{\eta}(\tilde{x}) = f(\tilde{x},\tilde{\sigma},\tilde{\tau})$ (where $\tilde{\sigma}$ and $\tilde{\tau}$ are also constants of integration). As we are considering the scaling transformation then $\tilde{\sigma} = \tilde{\eta}(0) = \lambda^{\beta}\eta(0) = \lambda^{\beta}\sigma$ and $\tilde{\tau} = \tilde{\eta}'(0) = \lambda^{\beta-1}\eta'(0) = \lambda^{\beta-1}\tau$. Also, we can say that

$$\phi(\tilde{x},\tilde{\eta},\tilde{\eta}') = \phi(\lambda x,\lambda^{\beta}\eta,\lambda^{\beta-1}\eta') = \lambda^{\beta}\phi(x,\eta,\eta') \tag{2.2.35}$$

and

$$\psi(\tilde{x},\tilde{\eta},\tilde{\eta}') = \psi(\lambda x,\lambda^{\beta}\eta,\lambda^{\beta-1}\eta') = \lambda^{\beta-1}\psi(x,\eta,\eta'). \tag{2.2.36}$$

Also, it is the case that as $\tilde{x} = \lambda x$ and $\tilde{\eta}(\tilde{x}) = \lambda^{\beta}\eta\left(\frac{\tilde{x}}{\lambda}\right)$ then

$$\Gamma\left(\frac{\tilde{\eta}(\tilde{x})}{\tilde{x}^{\beta}},\frac{\tilde{\eta}'(\tilde{x})}{\tilde{x}^{\beta-1}}\right) = \Gamma\left(\frac{\eta(x)}{x^{\beta}},\frac{\eta'(x)}{x^{\beta-1}}\right).$$

Hence

$$\begin{aligned}\theta(\tilde{x},\tilde{\sigma},\tilde{\tau}) &= \exp\left(\int^{\tilde{x}} \tilde{t}^{-1}\Gamma_{,2}\left(\frac{f(\tilde{t},\tilde{\sigma},\tilde{\tau})}{\tilde{t}^{\beta}},\frac{f_{,1}(\tilde{t},\tilde{\sigma},\tilde{\tau})}{\tilde{t}^{\beta-1}}\right)d\tilde{t}\right)\\ &= \exp\left(\int^{x} \lambda(\lambda t)^{-1}\Gamma_{,2}\left(\frac{f(t,\sigma,\tau)}{t^{\beta}},\frac{f_{,1}(t,\sigma,\tau)}{t^{\beta-1}}\right)dt\right)\end{aligned}$$

and so $\theta(\tilde{x},\tilde{\sigma},\tilde{\tau}) = \theta(x,\sigma,\tau)$. Also $\chi(\tilde{x},\tilde{\eta},\tilde{\eta}') = \chi(x,\eta,\eta')$. As $(\tilde{x},\tilde{\eta},\tilde{\eta}') =$

$(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta')$, and as $\chi(\tilde{x}, \tilde{\eta}, \tilde{\eta}') = \chi(x, \eta, \eta')$, we rewrite (2.2.34) as

$$\begin{aligned} M(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta') &= \frac{1}{\chi(x,\eta,\eta')}\Psi(\phi(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta'), \psi(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta')) \\ &= \frac{1}{\chi(x,\eta,\eta')}\Psi(\lambda^\beta \phi(x,\eta,\eta'), \lambda^{\beta-1}\psi(x,\eta,\eta')) \end{aligned}$$

by (2.2.35) and (2.2.36). As Ψ is an arbitrary function, it is possible to choose Ψ such that

$$\Psi(\lambda^\beta \phi(x,\eta,\eta'), \lambda^{\beta-1}\psi(x,\eta,\eta')) = \lambda^p \Psi(\phi(x,\eta,\eta'), \psi(x,\eta,\eta'))$$

for $p \in \mathbb{R}$.
Hence for $p \in \mathbb{R}$ it is possible to choose $M(\tilde{x}, \tilde{\eta}, \tilde{\eta}') = \hat{M}(\tilde{x}, \tilde{\eta}, \tilde{\eta}')$ such that

$$\hat{M}(\tilde{x}, \tilde{\eta}, \tilde{\eta}') = \frac{\lambda^p}{\chi(x,\eta,\eta')}\Psi(\phi(x,\eta,\eta'), \psi(x,\eta,\eta')) = \lambda^p \hat{M}(x,\eta,\eta') \qquad (2.2.37)$$

by (2.2.33). Now,

$$\hat{M}(x,\eta,\eta') = \hat{L}_{,33}(x,\eta,\eta') = \frac{\partial^2 \hat{L}}{\partial(\eta')^2}(x,\eta,\eta'). \qquad (2.2.38)$$

Also

$$\lambda^p \hat{M}(x,\eta,\eta') = \hat{M}(\tilde{x}, \tilde{\eta}, \tilde{\eta}') = \frac{\partial^2 \hat{L}}{\partial(\tilde{\eta}')^2}(\tilde{x}, \tilde{\eta}, \tilde{\eta}') = \frac{1}{\lambda^{2\beta-2}}\frac{\partial^2 \hat{L}}{\partial(\eta')^2}(\lambda x, \lambda^\beta \eta, \lambda^{\beta-1}\eta'). \qquad (2.2.39)$$

Hence (2.2.37) gives

$$\frac{\partial^2 \hat{L}}{\partial(\eta')^2}(\tilde{x}, \tilde{\eta}, \tilde{\eta}') = \lambda^{p+2(\beta-1)}\frac{\partial^2 \hat{L}}{\partial(\eta')^2}(x,\eta,\eta').$$

Now integration of (2.2.38) with respect to η' results in

$$\hat{L}(x,\eta,\eta') = P(x,\eta)\eta' + Q(x,\eta) + \left[\int^{\eta'}\left\{\int^{t}\hat{M}(x,\eta,s)\ ds\right\} dt\right] \qquad (2.2.40)$$

where P and Q are "constants" of integration. The Euler-Lagrange equation for

$$\mathcal{I}(\eta) = \int_0^b \hat{L}(x,\eta(x),\eta'(x))\ dx$$

for $\hat{L}$ of the form (2.2.40) is

$$\frac{\partial}{\partial\eta}\left[\int^{\eta'}\left\{\int^{t}\hat{M}(x,\eta,s)\;ds\right\}\;dt\right]-\frac{\partial}{\partial x}\left[\int^{\eta'}\hat{M}(x,\eta,s)\;ds\right]$$
$$-\frac{\partial}{\partial\eta}\left[\int^{\eta'}\hat{M}(x,\eta,s)\;ds\right]\eta'-\hat{M}(x,\eta,\eta')\eta''+\frac{\partial Q}{\partial\eta}(x,\eta)-\frac{\partial P}{\partial x}(x,\eta)=0, \tag{2.2.41}$$

where $\eta''=x^{\beta-2}\Gamma\left(\frac{\eta}{x^{\beta}},\frac{\eta'}{x^{\beta-1}}\right)$, and, as $P(x,\eta)$ and $Q(x,\eta)$ are arbitrary, $P(x,\eta)$ and $Q(x,\eta)$ are chosen such that $\hat{L}(x,\eta,\eta')$ satisfies

$$\hat{L}_{,2}(x,\eta,\eta')-\hat{L}_{,31}(x,\eta,\eta')-\eta'\hat{L}_{,32}(x,\eta,\eta')-\eta''\hat{L}_{,33}(x,\eta,\eta')=0, \tag{2.2.42}$$

where $\eta''=x^{\beta-2}\Gamma\left(\frac{\eta}{x^{\beta}},\frac{\eta'}{x^{\beta-1}}\right)$, and such that the Euler-Lagrange equation (2.2.41) is invariant under the set of scaling transformations H (as defined in §2.2.1), since that was our assumption. (Note that our choice of P and Q depends on the value of p.) Hence, P and Q must satisfy

$$P(\tilde{x},\tilde{\eta})=\lambda^{p+2\beta-1}P(x,\eta)\text{ and }Q(\tilde{x},\tilde{\eta})=\lambda^{p+3\beta-2}Q(x,\eta).$$

Hence, we have that $\hat{L}(\tilde{x},\tilde{\eta},\tilde{\eta}')=\lambda^{p+2(\beta-1)}\hat{L}(x,\eta,\eta')$, and so $\hat{L}$ is a relative invariant of H with multiplier λ^n (where $n=p+2(\beta-1)$) as required. □

Example 2.2.9 *Consider the second order differential equation*

$$\frac{d^2\eta}{dx^2}(x)+1=0. \tag{2.2.43}$$

This is invariant under the scaling transformation $\hat{\Pi}(\lambda,(x,\eta))=(\lambda x,\lambda^2\eta)$. *Hence*

$$\eta''+1=0 \tag{2.2.44}$$

is invariant under $\hat{\Pi}(\lambda,(x,\eta,\eta',\eta''))=(\lambda x,\lambda^2\eta,\lambda\eta',\eta'')$ *and thus (2.2.44) can be rewritten as*

$$\eta''=-1=\Gamma\left(\frac{\eta}{x^2},\frac{\eta'}{x}\right). \tag{2.2.45}$$

Thus $\Gamma_{,2}=0$ *and thus* $\chi(x,\eta,\eta')=m$, *a constant. Now we can integrate (2.2.43)* *twice to give*

$$\eta' = -x + \tau \text{ and } \eta = -\frac{x^2}{2} + \tau x + \sigma \tag{2.2.46}$$

which we can rearrange to get

$$\sigma = \eta - \frac{x^2}{2} + \eta' x \text{ and } \tau = \eta' + x. \tag{2.2.47}$$

Thus we have

$$mM(x, \eta, \eta') = \Psi\left(\eta - \frac{x^2}{2} + \eta' x, \eta' + x\right). \tag{2.2.48}$$

Thus if we put $\Psi(\sigma, \tau) = \tau^p$, $p \neq 0, \pm 1$, *then we have that* $L_{,33} = m(\eta' + x)^p$ *and thus* $L = \dfrac{m(\eta' + x)^{p+2}}{(p+1)(p+2)}$ *and the Euler-Lagrange equation becomes*

$$(\eta' + x)^p(\eta'' + 1) = 0, \tag{2.2.49}$$

with $P \equiv 0$, $Q \equiv 0$. *If* $\Psi(\sigma, \tau) = \tau$, *then* $L = m\left(\dfrac{(\eta')^3}{6} + \dfrac{x(\eta')^2}{2}\right)$ *and the Euler-Lagrange equation becomes* $m(\eta' + x)\eta'' + m\eta' - P_{,x} + Q_{,\eta} = 0$ *which becomes*

$$m(\eta' + x)(\eta'' + 1) = 0, \tag{2.2.50}$$

on putting $P \equiv 0$, $Q = mx\eta$ *or* $P = -\dfrac{mx^2}{2}$, $Q \equiv 0$, *for example. (Note again that the choice of* P *and* Q *does depend on the value of* p*.) Note that possible solutions of (2.2.49) and (2.2.50) are*

$$\eta(x) = -\frac{x^2}{2} + \tau x + \sigma \tag{2.2.51}$$

$$\text{and } \eta(x) = -\frac{x^2}{2} + \sigma. \tag{2.2.52}$$

As we have defined $\sigma = \eta(0)$ *and* $\tau = \eta'(0)$, $\eta(x)$ *given by (2.2.52) is an admissible solution if* $\eta'(0) = 0$ *(and is the same as (2.2.51)), and is not otherwise.*

2.2.3 Investigation of convexity following a change of co-ordinate system.

We assume throughout the rest of this chapter that $\tilde{L} > 0$ and that $\mathcal{I}(\eta)$ satisfies $(\mathcal{S}_n)$ as defined by Definition 2.2.1. Hence, by Proposition 2.2.3 we can write

$$L(x, \eta, \eta') = x^{n-1}\tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right).$$

From (B2) we can say that

$$\tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) \tag{2.2.53}$$

is also strictly convex in η' and convex in its second argument.

We are now in a position to determine the conditions under which convexity of the integrand under a change of coordinates occurs. Our approach will be using elements of Lie group theory. As mentioned in [52] Lie group theory has been used in considering the problem of integrating ordinary differential equations. Lie's "method" involves finding a symmetry group of an ordinary differential equation or a system of ordinary differential equations, with the observation that when an equation is invariant under a known group that equation can be integrated by a quadrature and the order of that equation is reduced by one. This is done by investigating the invariance properties of the differential equation (or system) concerned. Lie's fundamental observation was that if a sufficiently large group of symmetries is known the equation or system can be integrated by quadratures completely and a general solution can be found (see [16], [40] or [52]). This was by use of the invariance and infinitesimal generator results in §2.1. In this section we will make a change of variables, through applying some of those results in §2.1 to the case of a scaling transformation, and we will show that, under relative invariance of the integrand in the original coordinates, the integrand as expressed in the new coordinates is convex. We have noted already (see Proposition 2.1.32) that a change of variables can be made so that the original transformation in the original coordinates corresponds to a translation in the new coordinates. (In §2.2.4 we will investigate the uniqueness or otherwise of the solution to the variational problem of minimising $\mathcal{I}(\eta)$ as given by (2.2.13).)

The main result in this section is the following:

Proposition 2.2.10 *Let $\mathcal{I}(\eta)$ satisfy $(\mathcal{S}_n)$ and consider the set of scaling transformations $H := \{\hat{\Pi}(\lambda, (x, \eta)) : \lambda \in \mathbb{R}^+\}$ where $\hat{\Pi}(\lambda, (x, \eta)) = (\lambda x, \lambda^\beta \eta)$ and let*

$\eta(x)$ satisfy (B1) *and be such that $\dfrac{\eta(x)}{x^\beta}$ is invertible and strictly monotonic in $[0, b]$, and let L satisfy* (B2).

(1) Then there is a change of variables from (x, η) to (y, g) (with $y > 0$ and $g > 0$, g being the dependent variable with $g(d) = \hat{d}$ for all possible g) such that the integral can be rewritten as

$$\mathcal{I}(\eta) = \bar{\mathcal{I}}(g) = \int_c^d g'(y)\tilde{L}\left(y, n\frac{g(y)}{g'(y)} + \beta y\right) dy. \tag{2.2.54}$$

(2) Further, the integrand in $\bar{\mathcal{I}}(g)$ is convex in (g, g') where $g' \neq 0$.

Proof: The approach we will take in showing this result is the following:

1. We will derive the change of variables from (x, η) to (y, w) using Proposition 2.1.32 and derive the derivative term w'.
2. We will rewrite the integral in terms of the new coordinate system (y, w).
3. By making a second change of dependent variable from w to g, we will then show that under the hypotheses of relative invariance and strict convexity in η' of the original integrand that the resultant integrand is convex.

Step 1. We first derive the change of variables. We note that we can put $\chi = \eta'$ and $z = w'$ in Example 2.1.33 and from there we have that the coordinates (x, η, η') are related to (y, w, w') by

$$y = \frac{\eta}{x^\beta}, \; w = \log x \text{ and } w' = \frac{1}{x}\left\{\frac{\eta'}{x^{\beta-1}} - \frac{\beta\eta}{x^\beta}\right\}^{-1} = \frac{x^\beta}{x\eta' - \beta\eta}. \tag{2.2.55}$$

Step 2. Now, on returning to our integral (2.2.1), upon making a change of variables from (x, η, η') to (y, w, w') we can write

$$\tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) = \hat{L}(y, w, w'). \tag{2.2.56}$$

As $\tilde{L}$ is a function of the group invariants of H' for all $\lambda > 0$, $\tilde{\boldsymbol{v}}'(\tilde{L}) = 0$, and thus $\hat{\boldsymbol{v}}'(\hat{L}) = \dfrac{\partial \hat{L}}{\partial w} = 0$, and thus $\hat{L}$ is independent of w. As $w = \log x$ we can write

$$x^{n-1}\tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) = e^{(n-1)w}\hat{L}(y, w'). \tag{2.2.57}$$

Note that $e^{(n-1)w}\hat{L}(y, w')$ is not necessarily convex in (w, w').

Now, in considering the integral, we need to treat η as a function of x. Thus, we can write y as

$$y = \frac{\eta(x)}{x^\beta}. \tag{2.2.58}$$

and hence y is a function of x (and is strictly monotonic on $[0, b]$).
Also, if we assume that y given by (2.2.58) is an invertible function in $[0, b]$, then we can obtain x as a function of y. Hence as w is a function of x, we can obtain w as a function of y.
Now both

$$y = \frac{\eta(x)}{x^\beta} \qquad \text{and} \qquad w(y) = \log x$$

are functions of x and so we can consider $y(x)$ and $w(y(x))$. From the second equation we have

$$\frac{dw}{dy}\frac{dy}{dx} = \frac{1}{x}$$

which results in

$$\frac{dx}{dy} = e^w \frac{dw}{dy} \qquad \text{and} \qquad \frac{\eta'(x)}{x^{\beta-1}} = \frac{1}{w'(y)} + \beta y. \tag{2.2.59}$$

Integration of (2.2.57) with respect to x gives us that

$$\int_0^b x^{n-1}\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) dx = \int_0^b e^{(n-1)w(y(x))}\hat{L}\left(y(x), \frac{\frac{dw}{dx}(y(x))}{\frac{dy}{dx}(x)}\right) dx \tag{2.2.60}$$

which can be rewritten as

$$\int_0^b x^{n-1}\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) dx = \int_c^d e^{nw(y)}\tilde{L}\left(y, \frac{1}{w'(y)} + \beta y\right) w'(y)dy \tag{2.2.61}$$

due to (2.2.59). Therefore, we have that

$$\int_0^b L(x, \eta(x), \eta'(x))dx = \int_c^d e^{nw(y)}\tilde{L}\left(y, \frac{1}{w'(y)} + \beta y\right) w'(y)dy. \tag{2.2.62}$$

Step 3. Now if we put $ng(y) = e^{nw(y)}$, then $g'(y) = w'(y)e^{nw(y)}$ and (2.2.62) becomes

$$\mathcal{I}(\eta) = \int_0^b L(x, \eta(x), \eta'(x))dx = \int_c^d g'(y)\tilde{L}\left(y, n\frac{g(y)}{g'(y)} + \beta y\right) dy =: \bar{\mathcal{I}}(g) \tag{2.2.63}$$

which is (2.2.54). We have now shown (1) of Proposition 2.2.10.
In order to show (2) of Proposition 2.2.10, we note that, as $y = \frac{\eta(x)}{x^\beta}$ is a strictly monotonic function of x in $[0, b]$, $g' \neq 0$ in $[c, d]$. If $g' > 0$, then we can write $\bar{\mathcal{I}}(g)$ in the form in (2.2.63). If, on the other hand, $g' < 0$, then as $d < c$ we can write $\bar{\mathcal{I}}(g)$ as

$$\bar{\mathcal{I}}(g) = \int_d^c -g'(y)\tilde{L}\left(y, n\frac{g(y)}{g'(y)} + \beta y\right) dy. \tag{2.2.64}$$

We now consider

$$P(g, g') := g'\tilde{L}\left(y, n\frac{g}{g'} + \beta y\right) \tag{2.2.65}$$

where $g' > 0$.
In order to show that $P(g, g')$ is convex in (g, g') we require by Theorem 2.0.2 to show that

$$P(g, g') - P(h, h') \geq P_{,1}(h, h')(g - h) + P_{,2}(h, h')(g' - h') \tag{2.2.66}$$

Now $P_{,1}(h, h') = n\tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right)$ and

$$P_{,2}(h, h') = \tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) - n\frac{h}{h'}\tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right).$$

Thus for (2.2.66) to hold we must have

$$\begin{aligned}
&g'\tilde{L}\left(y, n\frac{g}{g'} + \beta y\right) - h'\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) \\
&\geq n\tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right)(g-h) \\
&\qquad\qquad + \left\{\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) - n\frac{h}{h'}\tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right)\right\}(g'-h') \\
&= \tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right)\left\{n\left(g - \frac{hg'}{h'}\right)\right\} + (g'-h')\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right).
\end{aligned} \tag{2.2.67}$$

However, using Theorem 2.0.2, and considering $P(g, g')$ as given by (2.2.65) with $g' > 0$, and $\tilde{L}$ convex in its second argument, we find that

$$\begin{aligned}
&g'\tilde{L}\left(y, n\frac{g}{g'} + \beta y\right) - h'\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) \\
&= g'\tilde{L}\left(y, n\frac{g}{g'} + \beta y\right) - g'\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) \\
&\qquad\qquad + g'\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) - h'\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) \\
&\geq g'\tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right)\left\{\frac{ng}{g'} + \beta y - \frac{nh}{h'} - \beta y\right\} + (g'-h')\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right) \\
&= \tilde{L}_{,2}\left(y, n\frac{h}{h'} + \beta y\right)\left\{n\left(g - \frac{hg'}{h'}\right)\right\} + (g'-h')\tilde{L}\left(y, n\frac{h}{h'} + \beta y\right).
\end{aligned} \tag{2.2.68}$$

Thus $P(g, g')$ is convex on the set $\{(g, g') \in \mathbb{R}^2 : g' > 0\}$. Similarly if we consider

$$Q(g, g') := -g'\tilde{L}\left(y, n\frac{g}{g'} + \beta y\right)$$

where $g' < 0$, it can be shown that $Q(g, g')$ is convex on the set $\{(g, g') \in \mathbb{R}^2 : g' < 0\}$. Hence, as $\tilde{L} > 0$ the integrand in $\bar{\mathcal{I}}(g)$ as given by (2.2.63) is the maximum of the two convex functions P and Q and is therefore convex in (g, g'), with $g' \neq 0$, as required. $\square$

Now, we recall that the motivation behind considering a change of variables was to see what can be said regarding uniqueness of solutions to classical minimisation problems. With this in mind, we now need to say something about the monotonicity of y as given by (2.2.58) in the case where η satisfies (B1). The next definition introduces Jacobi's notion of conjugate points.

Definition 2.2.11 **([18, Definition 5.7])** *Let $\eta_0 \in C^3([0,b])$ be an extremal for the variational problem of minimising $\mathcal{I}(\eta)$. If there is a function $\zeta \in C^2([0,c])$ which is a solution of*

$$\begin{aligned} \frac{d}{dx}\left[L_{,23}\left(x,\eta_0(x),\frac{d\eta_0}{dx}(x)\right)\zeta(x) + L_{,33}\left(x,\eta_0(x),\frac{d\eta_0}{dx}(x)\right)\frac{d\zeta}{dx}(x)\right] \\ -L_{,22}\left(x,\eta_0(x),\frac{d\eta_0}{dx}(x)\right)\zeta(x) - L_{,23}\left(x,\eta_0(x),\frac{d\eta_0}{dx}(x)\right)\frac{d\zeta}{dx}(x) = 0 \end{aligned} \tag{2.2.69}$$

where $\zeta(0) = \zeta(c) = 0$ and $\zeta(x)$ is not identically zero on any subinterval of $[0,c]$, then c is a ***conjugate point*** *of 0 corresponding to η_0.*

We refer to (2.2.69) as **Jacobi's equation** corresponding to η_0.
We note that as $\mathcal{I}(\eta)$ satisfies $(\mathcal{S}_n)$, by Remark 2.1.27 and Proposition 2.2.6 $\lambda^\beta \eta\left(\frac{x}{\lambda}\right)$ is a solution of the Euler-Lagrange equation whenever $\eta(x)$ is.
Let us therefore define the family of functions

$$\bar{\eta}(x,\lambda) := \lambda^\beta \eta\left(\frac{x}{\lambda}\right), \tag{2.2.70}$$

where λ is a constant (and $\lambda_0 = 1$). Clearly $\bar{\eta}(x,1) = \eta(x)$ and $\bar{\eta}(0,\lambda) = 0$ for all λ. Note also that the condition $\left.\frac{\partial \bar{\eta}}{\partial \lambda}(x,\lambda)\right|_{\lambda=1} = 0$ determines points on the curve $\mathcal{C}(\lambda) = \{(x,z) : z = \bar{\eta}(x,\lambda), x \in [0,b]\}$ which are conjugate to 0 (see [33], page 287).

Lemma 2.2.12 *Let $y = \frac{\eta(x)}{x^\beta}$ with $\eta(x)$ satisfying* (B1) *and define $\bar{\eta}(x,\lambda)$ by (2.2.70). Then y is strictly monotonic in $[0,b]$.*

Proof: Consider $y(x) = \frac{\eta(x)}{x^\beta}$ and suppose that $y(x)$ is not strictly monotonic in $[0,b]$. Then, there exists $x_0 \in (0,b)$ such that

$$\left.\frac{d}{dx}y(x)\right|_{x=x_0} = \frac{1}{x_0^\beta}\left(\frac{d\eta}{dx}(x_0) - \beta\frac{\eta(x_0)}{x_0}\right) = 0, \quad \text{and so } x_0\frac{d\eta}{dx}(x_0) - \beta\eta(x_0) = 0.$$

However,

$$\frac{\partial \bar{\eta}}{\partial \lambda}(x,1) = \beta\eta(x_0) - x_0\frac{d\eta}{dx}(x_0) = 0, \quad \text{and so } \left.\frac{\partial}{\partial \lambda}\left(\frac{\bar{\eta}(x,\lambda)}{x^\beta}\right)\right|_{\lambda=1} = 0,$$

and so there exists a conjugate point $x = x_0$ to $x = 0$, which we cannot have, as $\eta(x)$ is a minimiser of $\mathcal{I}(\eta)$ (see [33]). Thus it follows that

$$y(x) = \frac{\eta(x)}{x^\beta}$$

must be strictly monotonic in $[0, b]$. $\square$

2.2.4 Investigation of the uniqueness of the solution.

We now consider the question of uniqueness of solutions. Here we will show uniqueness in certain cases (such as the brachistochrone problem and Dido's problem), as well as highlighting the problems in showing uniqueness in other cases (such as Zenodoros' problem). The approach we will take in attempting to do this is based on the proof of [60, Theorem 3.8].

Suppose that there exist two solutions $\eta_i \in C^2((0,b)) \cup C^0([0,b])$, $i = 1, 2$, $\eta_1(x) \not\equiv \eta_2(x)$, satisfying $\eta_1(0) = \eta_2(0) = 0$, $\eta_1(b) = \eta_2(b)$, and let g_i, $i = 1, 2$, be the corresponding function after the change of dependent and independent variables from (x, η) to $(y, g) = \left(\frac{\eta(x)}{x^\beta}, \frac{x^n}{n}\right)$, g being a function of y.

As $y = \frac{\eta(x)}{x^\beta}$, and as η_1 and η_2 must both satisfy $\eta_1(0) = \eta_2(0) = 0$, then it is possible that $\frac{\eta_1(x)}{x^\beta}$ and $\frac{\eta_2(x)}{x^\beta}$ could tend to different values as x tends to zero, and thus different solutions could give rise to different intervals on which y is defined.

As a consequence, we will consider the problem for two solutions η_1 and η_2 which satisfy

$$\lim_{x \to 0} \frac{\eta_1(x)}{x^\beta} = \lim_{x \to 0} \frac{\eta_2(x)}{x^\beta}, \tag{2.2.71}$$

and so the interval for y is fixed. (The above equation is true for the brachistochrone problem and Dido's problem.)

As shown already, the integral that we are considering is

$$\mathcal{I}(\eta_i) = \bar{\mathcal{I}}(g_i) = \int_c^d g_i'(y) \tilde{L}\left(y, n\frac{g_i(y)}{g_i'(y)} + \beta y\right) dy, \qquad i = 1, 2. \tag{2.2.72}$$

Put $P(g, g')$ as in (2.2.65). Then, as the right-hand side of (2.2.65) is strictly convex in (g_i, g_i'), we can use the argument of the proof of Theorem 2.0.3 to say that

$$\begin{aligned}
&\int_c^d P(g_1(y), g_1'(y))dy \\
&\geq \int_c^d P(g_2(y), g_2'(y))dy + \{(g_1 - g_2)P_{,2}(g_2, g_2')\}|_c^d \\
&\quad + \int_c^d \left\{P_{,1}(g_2(y), g_2'(y)) - \frac{d}{dy}\left(P_{,2}(g_2(y), g_2'(y))\right)\right\}(g_1(y) - g_2(y))dy
\end{aligned} \tag{2.2.73}$$

with equality if and only if $g_1(y) \equiv g_2(y)$, since we are dealing with functions that are equal on the boundary. As we are assuming that $\eta_1(x) \not\equiv \eta_2(x)$ then $g_1(y) \not\equiv g_2(y)$, and so we have that (2.2.73) must hold strictly.
Since we are assuming that $\eta_i(x)$ is a solution of the Euler-Lagrange equation for $\mathcal{I}(\eta)$, then $g_i(y)$ is a solution of the Euler-Lagrange equations for $\bar{\mathcal{I}}(g)$. (See [20, §2.4 B] for details about the invariant nature of the Euler-Lagrange equations.) Thus, as

$$\int_c^d \left\{P_{,1}(g_2(y), g_2'(y)) - \frac{d}{dy}\left(P_{,2}(g_2(y), g_2'(y))\right)\right\}(g_1(y) - g_2(y))dy = 0,$$

then (2.2.73) becomes

$$\int_c^d P(g_1(y), g_1'(y))dy > \int_c^d P(g_2(y), g_2'(y))dy + [(g_1 - g_2)P_{,2}(g_2, g_2')]|_c^d. \tag{2.2.74}$$

Now in order to continue with showing uniqueness, we need to know something about the behaviour of $P_{,2}(g_i, g_i')$ $(i = 1,\ 2)$ at the end-points $y = c$, $y = d$. In particular we want to see if $P_{,2}(g_i, g_i')$ is finite at c and d, since if it is then we can relate $\mathcal{I}(\eta_1)$ and $\mathcal{I}(\eta_2)$ by an inequality.
At this point we will consider examples (I) and (II). For the brachistochrone problem ([68, §3]: here $\beta = 1$ and $n = \frac{1}{2}$) and Dido's problem ([68, §4]: here $\beta = 1$ and $n = 1$), it can be shown, by considering the first integral (since the integrand in each case is explicitly independent of x), that $P_{,2}(g, g')$ is finite as y tends to each of the end-points $y = c$ (corresponding to $x = 0$) and $y = d$ (corresponding to $x = b$: in fact, since $\eta(b) = 0$ in both these examples, $d = 0$). Hence, in these two cases

$$\{(g_j - g_i)P_{,2}(g_i, g_i')\}|_c^d = 0, \qquad i, j = 1, 2,\ i \neq j, \tag{2.2.75}$$

and so we have that

$$\mathcal{I}(\eta_1) = \bar{\mathcal{I}}(g_1) = \int_c^d P(g_1(y), g_1'(y))dy > \int_c^d P(g_2(y), g_2'(y)) = \bar{\mathcal{I}}(g_2) = \mathcal{I}(\eta_2), \tag{2.2.76}$$

and we can interchange g_1 and g_2, and thus we have uniqueness for these two cases, and for any case where (2.2.71) and (2.2.75) hold. We note also that

$$\begin{aligned} P_{,2}(g, g') &= \tilde{L}\left(y, n\frac{g}{g'} + \beta y\right) - \frac{ng}{g'}\tilde{L}_{,2}\left(y, n\frac{g}{g'} + \beta y\right) \\ &= \tilde{L}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right) + \left(\frac{\beta\eta}{x^\beta} - \frac{\eta'}{x^{\beta-1}}\right)\tilde{L}_{,2}\left(\frac{\eta}{x^\beta}, \frac{\eta'}{x^{\beta-1}}\right). \end{aligned} \tag{2.2.77}$$

Hence, we can say the following:

Proposition 2.2.13 *Let $\mathcal{I}(\eta)$ be of the form*

$$\mathcal{I}(\eta) = \int_0^b L(x, \eta(x), \eta'(x))\, dx$$

and $\mathcal{A}$ be the set of admissible functions where

$$\mathcal{A} = \{\eta \in C^2((0,b)) \cup C^0([0,b]) : \eta(0) = 0, \eta(b) = \hat{b}, \eta(x) > 0\}$$

and suppose that $L_{,33} > 0$ and that $\mathcal{I}(\eta)$ satisfies $(\mathcal{S}_n)$.
Suppose also that for any solution $\eta(x)$ of the Euler-Lagrange equation, $\dfrac{\eta(x)}{x^\beta}$ is monotonic in $[0,b]$, $\lim\limits_{x\to 0}\dfrac{\eta(x)}{x^\beta}$ is the same for all solutions and

$$\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) + \left(\frac{\beta\eta(x)}{x^\beta} - \frac{\eta'(x)}{x^{\beta-1}}\right)\tilde{L}_{,2}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)$$

is finite in $[0,b]$. Then there exists at most one minimiser of $\mathcal{I}(\eta)$ on $\mathcal{A}$.

We can also determine the value of $\mathcal{I}(\eta)$ explicitly for the cases where $P_{,2}(g, g')$ is finite at c and d. As (2.2.77) holds, then the behaviour of $P_{,2}(g, g')$ at the end-points will also determine what we can say about the value of $\mathcal{I}(\eta)$ at a minimiser, by (2.2.15). If $P_{,2}(g, g')$ is finite as y tends to the end-points then

$$\lim_{x\to 0}\frac{x^n}{n}\left[\tilde{L}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right) + \left(\frac{\beta\eta(x)}{x^\beta} - \frac{\eta'(x)}{x^{\beta-1}}\right)\tilde{L}_{,2}\left(\frac{\eta(x)}{x^\beta}, \frac{\eta'(x)}{x^{\beta-1}}\right)\right] = 0.$$

By Remark 2.2.5, we can write $\mathcal{I}(\eta)$ in the form (2.2.15), and so, for these two cases, we can write

$$\mathcal{I}(\eta) = \frac{b^n}{n}\left[\tilde{L}\left(\frac{\eta(b)}{b^\beta}, \frac{\eta'(b)}{b^{\beta-1}}\right) + \left(\frac{\beta\eta(b)}{b^\beta} - \frac{\eta'(b)}{b^{\beta-1}}\right)\tilde{L}_{,2}\left(\frac{\eta(b)}{b^\beta}, \frac{\eta'(b)}{b^{\beta-1}}\right)\right]. \quad (2.2.78)$$

Thus we have an explicit form for $\mathcal{I}(\eta)$.

Discussion.

In trying to show uniqueness in a more general setting, problems still remain. In particular, as already mentioned, two different solutions could result in different intervals for the independent variable y, the example of Zenodoros' problem being one such case. In the examples of the brachistochrone problem and Dido's problem, the scaling transformation is $\hat{\Pi}(\lambda, (x, \eta)) = (\lambda x, \lambda\eta)$, and it can be shown that the interval for y is fixed. Also, for these two examples, the first integral of the Euler-Lagrange equation is used to show various properties that are required, such as $P_{,2}(g, g')$ being finite at c and d. It is not clear how this could be shown in general, particularly for integrals where there is explicit dependence in x. Another example where the relative invariance of the integrand for a scaling transformation is considered is given by Ball & Mizel ([12, §3]). In that example they consider the problem of minimising

$$\mathcal{I}(\eta) = \int_0^1 (x^2 - \eta^3)^2(\eta')^{14} + \kappa(\eta')^2 \, dx$$

subject to $\eta(0) = 0$, $\eta(1) = k$. There, the integrand is relative invariant under the extended scaling transformation $\hat{\Pi}(\lambda, (x, \eta, \eta')) = (\lambda x, \lambda^{\frac{2}{3}}\eta, \lambda^{-\frac{1}{3}}\eta')$. However, depending on the value of κ there can be two, one or no solutions of the Euler-Lagrange equations, and for κ sufficiently small and k sufficiently large the minimiser does not satisfy the Euler-Lagrange equation.

Another example is that of determining the minimum area of revolution. There, the integral is

$$\hat{\mathcal{I}}(\chi) = -\int_0^1 \chi(x)\sqrt{(1 - (\chi'(x))^2)} \, dx$$

and the boundary conditions are $\chi(0) = 1$, $\chi(1) = k$. We can satisfy the condition $\eta(0) = 0$ by putting $\eta(x) = \chi(x) - 1$ but the integral becomes

$$\mathcal{I}(\eta) = -\int_0^1 (\eta(x) - 1)\sqrt{(1 - (\eta'(x))^2)} \, dx$$

and thus $\mathcal{I}(\eta)$ does not satisfy the condition $(\mathcal{S}_n)$, since the integrand is not relative invariant under the scaling transformation

$$\Pi(\varepsilon, (x, \eta, \eta')) = (e^{\varepsilon}x, e^{\beta\varepsilon}\eta, e^{(\beta-1)\varepsilon}\eta')$$

for any value of β. We note that a solution of the Euler-Lagrange equation is of the form $\chi(x) = (\cosh^{-1}(d))^{-1}\cosh(x\cosh^{-1}(d) + d)$, and there can be either no, one or two possible values of d, depending on k, and if there are two, it can be the case that there exists $t \in [0, 1]$ such that

$$\eta'(x) - \beta\frac{\eta(x)}{x} = 0 \tag{2.2.79}$$

holds. (See, e.g., [18] for full details of that problem.)

Chapter 3

Nonlinear elasticity.

In the rest of this work we will be considering variational problems in two-dimensional nonlinear elasticity. In order to consider these type of variational problems we will require key results and concepts from the theory of Sobolev spaces, which in turn require key results from functional analysis. These will be presented in §3.1, along with a brief discussion of direct methods in the calculus of variations. In §3.2 we give a basic introduction to nonlinear elasticity.

3.1 Basic analytic results.

In order to study higher-dimensional variational problems we need some basic concepts and results from functional analysis. For more details and proofs, see any basic introduction to functional analysis, such as [15], [43] or [71]. We refer to [2] or [27] for results on L^p and Sobolev spaces. Throughout this section we will let Ω be a open (but not necessarily bounded) set in $\mathbb{R}^n$ unless otherwise stated.

We first introduce the concept of weak convergence in the following definition.

Definition 3.1.1 ([15] page 31 or [71] page 49) *Let X be a normed vector space and let X^* be its dual space (the space of all linear continuous functionals on X). A sequence $(x_n) \subset X$* ***converges weakly*** *to $x \in X$, denoted by $x_n \rightharpoonup x$, if $\Phi(x_n) \to \Phi(x)$ for every $\Phi \in X^*$ as $n \to \infty$.*

In order to avoid any confusion convergence in norm will be referred to henceforth as **strong convergence**.

We note the following properties of weakly convergent sequences.

Theorem 3.1.2 *Any weakly convergent sequence in a normed vector space is bounded.*

In order to present the next property we require the following notion of a uniformly convex space.

Definition 3.1.3 ([2] page 7) *A norm on a space X is said to be* ***uniformly convex*** *if for every number ε satisfying $0 < \varepsilon \leq 2$ there exists a number $\delta(\varepsilon) > 0$ such that if $x,\, y \in X$ satisfy $\| x \|_X = \| y \|_X = 1$ and $\| x - y \|_X \geq \varepsilon$, then*

$$\left\| \frac{x+y}{2} \right\|_X \leq 1 - \delta(\varepsilon).$$

The normed space X is said to be uniformly convex if its norm is uniformly convex.

Theorem 3.1.4 ([15] page 31) *Let X be a uniformly convex Banach space with norm $\| \cdot \|_X$. If $x_n \rightharpoonup x$ in X and $\| x_n \|_X \to \| x \|_X$ then $x_n \to x$ in X.*

The dual space of X^*, denoted X^{**}, is called the second dual of X.

Definition 3.1.5 ([43] pp 107-8) *The mapping $x \to \hat{x}$ from X to X^{**} where $\hat{x}$ is defined by $\hat{x}(\Phi) = \Phi(x)$ for all $\Phi \in X^*$ is called the* ***canonical embedding*** *of X in X^{**}.*

The canonical embedding of X in X^{**} is a norm-preserving, linear, one-to-one mapping of X into X^{**} and the image of X is denoted by $\hat{X}$. It is also isometric.

Definition 3.1.6 *If $\hat{X} = X^{**}$ then X is a* ***reflexive space****.*

One important property of reflexive spaces is the following result on bounded sequences.

Theorem 3.1.7 (Banach) *Every bounded sequence in a reflexive space has a weakly convergent subsequence.*

(See, e.g., [71, §2.8, Proposition 6] page 64.)
We also introduce the concept of sequential weak compactness, which will be useful in considering variational problems in infinite-dimensional spaces.

Definition 3.1.8 ([31, Definition 4.10.2] page 161) *Let X be a normed vector space. A set $M \subset X$ is* ***sequentially weakly compact*** *if every sequence (x_n) in M contains a subsequence that converges weakly to a point in M.*

We now introduce the concept of coercivity, which will be useful in obtaining boundedness of minimising sequences. (Here $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$.)

Definition 3.1.9 *Let X be a normed vector space with norm $\| \cdot \|_X$. A function $f : X \to \bar{\mathbb{R}}$ is said to be* **coercive** *if and only if $f(x) \to \infty$ as $\| x \|_X \to \infty$.*

We next introduce the notions of lower semicontinuity and sequential weak lower semicontinuity.

Definition 3.1.10 *Let X be a normed vector space.*

(i) A function $f : X \to \bar{\mathbb{R}}$ is said to be **lower semicontinuous** *if and only if the set $\{x \in X : f(x) \leq \alpha\}$ is closed for each α;*

(ii) A function $f : X \to \bar{\mathbb{R}}$ is said to be **sequentially weakly lower semicontinuous** *if for every sequence $(x_n) \subset X$ converging weakly to $x \in X$ we have $f(x) \leq \liminf_{n\to\infty} f(x_n)$.*

We now present a key minimisation result which is applied in the direct method of the calculus of variations.

Theorem 3.1.11 *Let X be a reflexive Banach space and let $f : X \to \bar{\mathbb{R}}$ be coercive and sequentially weakly lower semicontinuous. Then f attains a minimum on X.*

Proof: Let $m = \inf_{x\in X} f(x)$ and without loss of generality assume $m \neq \infty$ (since if $m = \infty$ then $f \equiv \infty$ and the result follows trivially). Choose $M > m$. By coercivity of f there exists $d > 0$ such that $f(x) \geq M$ for all $\| x \|_X \geq d$. Now let $(x_n) \in X$ be a minimising sequence for f on X, that is $f(x_n) \to m$ as $n \to \infty$. Hence, $f(x_n) \leq M$ for all sufficiently large n and so $\| x \|_X \leq d$ for all n sufficiently large. Since X is reflexive, there exists a weakly convergent subsequence $(x_{n_j}) \in X$ converging weakly to $x_0 \in X$ as $j \to \infty$ by Theorem 3.1.7. Hence, as f is sequentially weakly lower semicontinuous

$$f(x_0) \leq \liminf_{j\to\infty} f(x_{n_j}) = m = \inf_{x\in X} f(x)$$

and thus $f(x_0) = \inf_{x\in X} f(x)$ as required. □

We now give sufficient conditions for f to be sequentially weakly lower semicontinuous. This will prove useful in the context of variational problems in nonlinear elasticity.

Lemma 3.1.12 *Let X be a real normed vector space and let $f : X \to \bar{\mathbb{R}}$ be lower semicontinuous and convex. Then f is sequentially weakly lower semicontinuous.*

Theorem 3.1.13 *Let X be a reflexive Banach space and let $f : X \to \bar{\mathbb{R}}$ be a convex, lower semicontinuous and coercive function. Then f attains a minimum on X.*

Proof: This is a consequence of Theorem 3.1.11 and Lemma 3.1.12. □

L^p-spaces and Sobolev spaces.

We now introduce the Lebesgue spaces $L^p(\Omega)$ $(1 \leq p \leq \infty)$.

Definition 3.1.14 ([27] page 17) *Let $1 \leq p < \infty$. A measurable function $f : \Omega \to \mathbb{R}$ is said to be in $L^p(\Omega)$ if*

$$\| f \|_p := \left(\int_\Omega |f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

If $p = \infty$, then f is said to be in $L^\infty(\Omega)$ if $\| f \|_\infty = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)| < \infty$ where $\operatorname*{ess\,sup}_{x \in \Omega} |f(x)| = \inf\{K : |f(x)| \leq K \text{ a.e. in } \Omega\}$.

Theorem 3.1.15 ([2, Theorem 2.35] page 31) *For $1 < p < \infty$ the space $L^p(\Omega)$ is reflexive.*

We next introduce the notion of local integrability. In the following, we say that $A \subset\subset \Omega$ if $\bar{A} \subset \Omega$ and $\bar{A}$ is compact.

Definition 3.1.16 ([2] page 19) *A measurable function f on Ω is said to be* ***locally integrable*** *if f is Lebesgue integrable on every set $A \subset\subset \Omega$. We denote this by $f \in L^1_{\text{loc}}(\Omega)$.*

As noted in [2], corresponding to every $f \in L^1_{\text{loc}}(\Omega)$ there is a distribution Ψ in the dual space of $C_0^\infty(\Omega)$ defined by

$$\Psi(\phi) = \int_\Omega f(x)\phi(x) \, dx$$

for $\phi \in C_0^\infty(\Omega)$, where $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω.

Also, as noted in [2], a sequence (ϕ_n) of functions in $C_0^\infty(\Omega)$ *converges in the sense of the space of test functions $\mathcal{D}(\Omega)$* to $\phi \in C_0^\infty(\Omega)$ provided that there exists $K \subset\subset \Omega$ such that the support of $(\phi_n - \phi)$ is a subset of K for every n, and $\lim_{n \to \infty} D^\alpha \phi_n(x) = D^\alpha \phi(x)$ uniformly on K for each multi-index α, where

$$D^\alpha \phi(x) = \frac{\partial^{|\alpha|}\phi(x)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$$

and where $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

The dual space $\mathcal{D}'(\Omega)$ of $\mathcal{D}(\Omega)$ is called the *space of distributions*, and it is given the weak-* topology as dual of $\mathcal{D}(\Omega)$. For a more detailed discussion on the space of test functions and the space of distributions, see [59].

We now define the conjugate exponent of p. This will be useful in expressing the dual space of $L^p(\Omega)$. For $p = \infty$, we put $\frac{1}{p} = 0$.

Definition 3.1.17 *For $1 < p < \infty$ we denote the* ***conjugate exponent*** *of p by p' where p' is such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $p = 1$, we put $p' = \infty$ and vice versa.*

Let $1 \leq p \leq \infty$ and let p' denote the conjugate exponent of p. For each element $g \in L^{p'}(\Omega)$ we can define a linear functional Φ on $L^p(\Omega)$ by

$$\Phi(f) = \int_\Omega f(x)g(x)\ dx \tag{3.1.1}$$

for $f \in L^p(\Omega)$ (see, e.g., [2, 2.30] page 38). We now present the Riesz representation theorem for $L^p(\Omega)$ spaces where $1 \leq p < \infty$. This shows how elements in the dual space of $L^p(\Omega)$ correspond to functions in $L^{p'}(\Omega)$. One consequence of this is that every continuous linear functional on $L^p(\Omega)$ is of the form Φ given by (3.1.1) for some $g \in L^{p'}(\Omega)$.

Theorem 3.1.18 (Riesz representation theorem) *Let $1 \leq p < \infty$ and let Φ be a bounded linear functional on $L^p(\Omega)$. Then there exists $g \in L^{p'}(\Omega)$ such that for all $f \in L^p(\Omega)$*

$$\Phi(f) = \int_\Omega f(x)g(x)\ dx.$$

(See, e.g., [2, Theorem 2.33].)

We now give two weak convergence results, both following as consequences of the Riesz representation theorem. The first is the notion of weak convergence in L^p spaces.

Corollary 3.1.19 ([27] page 17) *For $1 \leq p < \infty$, $f_n \rightharpoonup f$ in $L^p(\Omega)$ if*

$$\int_\Omega f_n(x)g(x)\ dx \to \int_\Omega f(x)g(x)\ dx$$

for every $g \in L^{p'}(\Omega)$.

The second convergence result will be used in proving weak continuity results later in this thesis.

Proposition 3.1.20 *For $1 < p < \infty$ let $f_n \to f$ in $L^p(\Omega)$ and $g_n \rightharpoonup g$ in $L^{p'}(\Omega)$. Then $f_n g_n \rightharpoonup fg$ in $L^1(\Omega)$.*

Proof: Let $\phi \in L^\infty(\Omega)$. Then we have

$$\int_\Omega \phi(x) f_n(x) g_n(x)\ dx = \int_\Omega \phi(x)(f_n(x) - f(x)) g_n(x)\ dx + \int_\Omega \phi(x) f(x) g_n(x)\ dx.$$

Since there exists m such that $|\phi(x)| \le m$ for a.e. x in Ω, by Hölder's inequality we can say that

$$\begin{aligned}\int_\Omega |\phi(x)(f_n(x) - f(x)) g_n(x)|\ dx &\le m \int_\Omega |f_n(x) - f(x)||g_n(x)|\ dx \\ &\le m \parallel f_n - f \parallel_p \parallel g_n \parallel_{p'} \to 0\end{aligned}$$

as $n \to \infty$, since the sequence (g_n) is bounded in $L^{p'}(\Omega)$ by Theorem 3.1.2. Also $\phi f \in L^p(\Omega)$ so that

$$\int_\Omega \phi(x) f(x) g_n(x)\ dx \to \int_\Omega \phi(x) f(x) g(x)\ dx$$

as $n \to \infty$ and thus

$$\int_\Omega \phi(x) f_n(x) g_n(x)\ dx \to \int_\Omega \phi(x) f(x) g(x)\ dx$$

as $n \to \infty$. $\square$

We now state the following result on sequential weak lower semicontinuity, which will be useful in showing the existence of minimisers in nonlinear elasticity.

Theorem 3.1.21 ([11, Theorem 5.4]) *Let $\Phi : \Omega \times \mathbb{R}^q \times \mathbb{R}^\sigma \to \mathbb{R}$ satisfy the following properties:*

1. *$\Phi(\cdot, y, z) : \Omega \to \bar{\mathbb{R}}$ is measurable for every $(y, z) \in \mathbb{R}^q \times \mathbb{R}^\sigma$,*
2. *$\Phi(x, \cdot, \cdot) : \mathbb{R}^q \times \mathbb{R}^\sigma \to \bar{\mathbb{R}}$ is continuous for almost all $x \in \Omega$,*
3. *$\Phi(x, y, \cdot) : \mathbb{R}^\sigma \to \bar{\mathbb{R}}$ is convex for almost all $x \in \Omega$ and all $y \in \mathbb{R}^q$.*

Let y_n, $y : \mathbb{R}^q \to \bar{\mathbb{R}}$ be measurable functions such that $y_n \to y$ almost everywhere, and let $z_n \rightharpoonup z$ in $(L^1(\Omega))^\sigma$ as $n \to \infty$. Suppose further that there exists $\zeta \in L^1(\Omega)$ such that $\Phi(x, y_n(x), z_n(x)) \ge \zeta(x)$ and $\Phi(x, y(x), z(x)) \ge \zeta(x)$ for all n and *almost all* $x \in \Omega$. *Then*

$$\int_\Omega \Phi(x, y(x), z(x))\ dx \le \liminf_{n\to\infty} \int_\Omega \Phi(x, y_n(x), z_n(x))\ dx.$$

We now present the fundamental lemma of the calculus of variations. This is used in obtaining the Euler-Lagrange equations satisfied by a minimiser of an integral functional.

Lemma 3.1.22 (Fundamental lemma of the calculus of variations) *Let* $f \in L^1_{\rm loc}(\Omega)$ *be such that*

$$\int_\Omega f(x)\phi(x)\ dx = 0$$

for all $\phi \in C_0^\infty(\Omega)$. *Then* $f = 0$ *a.e. in* Ω.

The proof of this uses mollifiers: see [2, Lemma 3.26] page 59.
We now introduce the concept of Sobolev spaces. Sobolev spaces have become a useful tool in the study of partial differential equations and variational problems. In order to introduce these spaces, we require the notion of a weak derivative.

Definition 3.1.23 *Let* f *be locally integrable in* $\Omega \subset \mathbb{R}^n$ *and* $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ *be any multi-index. A locally integrable function* g *is the* α^{th}***-weak derivative*** *of* f *if it satisfies*

$$\int_\Omega \phi(x)g(x)\ dx = (-1)^{|\alpha|} \int_\Omega f(x)D^\alpha\phi(x)\ dx$$

for all $\phi \in C_0^\infty(\Omega)$.

Definition 3.1.24 *Let* $s \in \mathbb{N}$ *and* $1 \le p \le \infty$. *Then we define*

$$W^{s,p}(\Omega) := \{f : f \in L^p(\Omega), D^\alpha f \in L^p(\Omega)\ \textit{for}\ |\alpha| \le s\}$$

where $D^\alpha f$ *denotes the* α^{th}*-weak derivative of* f. *To this space* $W^{s,p}(\Omega)$ *is associated the norm*

$$\| f \|_{s,p} = \left(\sum_{0\le|\alpha|\le s} \| D^\alpha f \|_p^p \right)^{\frac{1}{p}}$$

if $1 \le p < \infty$, *with* $\| f \|_{s,\infty} = \max_{0\le|\alpha|\le s} \{\| D^\alpha f \|_\infty\}$ *if* $p = \infty$.

Essentially the Sobolev space $W^{s,p}(\Omega)$ consists of functions in $L^p(\Omega)$ for which all weak partial derivatives of order less than or equal to s also belong to $L^p(\Omega)$. We now present some fundamental properties of Sobolev spaces.

Theorem 3.1.25 ([2, Theorem 3.2 & Theorem 3.5]) *$W^{s,p}(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$. If $1 < p < \infty$ then $W^{s,p}(\Omega)$ is also reflexive.*

We also note the following density result, which will be of use later.

Theorem 3.1.26 *$C^\infty(\Omega)$ is dense in $W^{s,p}(\Omega)$ for all $1 \leq p < \infty$.*

The proof of this uses mollifiers: see [2] pp 51-56.
The next result we present is the Rellich-Kondrachov theorem for Sobolev spaces $W^{1,p}(\Omega)$. This result will prove useful in showing sequential weak lower semi-continuity of integral functionals. The Rellich-Kondrachov theorem is concerned with compact embeddings of $W^{s,p}(\Omega)$ into $L^q(\Omega)$. Since we are mainly considering Sobolev spaces $W^{1,p}(\Omega)$ throughout the rest of this work, we will present this theorem for such spaces only. In order to do this we require the definition of a embedding. (In the three definitions that follow we are considering general normed vector spaces.)

Definition 3.1.27 ([2] page 9) *The normed vector space X with norm $\|\cdot\|_X$ is* ***embedded*** *in the normed vector space Y with norm $\|\cdot\|_Y$, denoted by $X \hookrightarrow Y$, provided that X is a vector subspace of Y and the operator ι defined on X into Y by $\iota(x) = x$ for all $x \in X$ is continuous.*

Alternatively, X is embedded in Y if X is a vector subspace of Y and (strong) convergence of a sequence (x_n) in X implies (strong) convergence of (x_n) in Y (see [15] page 26). We also introduce the notion of a compact linear operator.

Definition 3.1.28 ([43, Definition 8.1-1] pp 405-6) *Let X and Y be normed vector spaces. An operator $T : X \to Y$ is called a* ***compact linear operator*** *if T is linear and if for every bounded subset M of X, $T(M)$ is relatively compact, that is $\overline{T(M)}$ is compact.*

Thus, we now define a compact embedding.

Definition 3.1.29 *A normed vector space X with norm $\|\cdot\|_X$ is* ***compactly embedded*** *in a normed vector space Y with norm $\|\cdot\|_Y$ if X is embedded in Y and if the continuous injection $\iota : X \to Y$ is a compact linear operator.*

Alternatively, X is compactly embedded in Y if ι maps each bounded sequence (x_n) in X (with norm $\|\cdot\|_X$) into a sequence $(\iota(x_n))$ that contains a subsequence converging to some limit in Y (with norm $\|\cdot\|_Y$). We also require the notion of a Lipschitz boundary for a bounded open set.

Definition 3.1.30 *Let Ω be a bounded open set. Then Ω has a **Lipschitz boundary** if the boundary of Ω is locally the graph of a Lipschitz function.*

Theorem 3.1.31 (Rellich-Kondrachov theorem) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $1 \leq p \leq \infty$.*

1. *If $1 \leq p < n$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q \leq \dfrac{np}{n-p}$ and the embedding is compact for every $1 \leq q < \dfrac{np}{n-p}$.*
2. *If $p = n$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for every $1 \leq q < \infty$ and the embedding is compact.*
3. *If $p > n$, then $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ and the embedding is compact.*

From the above we can say the following.

Corollary 3.1.32 ([27] page 26) *Let $f_k \rightharpoonup f$ in $W^{1,p}(\Omega)$.*

1. *If $1 \leq p < n$, then $f_k \to f$ in $L^q(\Omega)$ for $1 \leq q < \dfrac{np}{n-p}$.*
2. *If $p = n$, then $f_k \to f$ in $L^q(\Omega)$ for $1 \leq q < \infty$.*
3. *If $p > n$, then $f_k \to f$ in $C(\bar{\Omega})$.*

Remark 3.1.33 *Note in particular that for $n = 2$ (or 3) **weak** convergence in $W^{1,2}(\Omega)$ implies **strong** convergence in $L^2(\Omega)$. Thus if we can show that a sequence is bounded in $W^{1,2}(\Omega)$ then there exists a subsequence that is **weakly** convergent in $W^{1,2}(\Omega)$ by Banach's result (Theorem 3.1.7) and **strongly** convergent in $L^2(\Omega)$ by the Rellich-Kondrachov Theorem (Theorem 3.1.31). (This will prove useful when considering sequences in $W^{1,1}(\Omega)$, as $W^{1,1}(\Omega)$ is not a reflexive space.)*

The question of existence of solutions to the problem of minimising a given functional.

A central problem in the calculus of variations is to prove the existence of functions which minimise a given integral functional on a given admissible set subject to prescribed boundary conditions. Consider functionals of the form

$$\mathcal{J}(f) = \int_\Omega G(x, f(x), \nabla f(x)) \, \mathrm{dx}$$

on the set of admissible functions $\mathcal{A} = \{f \in W^{1,1}(\Omega) : f = f_0 \text{ on } \partial\Omega\}$, where f_0 is given and the boundary conditions are understood in terms of trace. Here $\Omega \subset \mathbb{R}^n$ is an open bounded set and $f : \Omega \to \mathbb{R}^m$. We associate ∇f with the $m \times n$ Jacobian matrix of its partial derivatives, and so $\nabla f : \Omega \to M^{m\times n}$. We will also assume that $G : \Omega \times \mathbb{R}^m \times M^{m\times n} \to \mathbb{R}$ is a continuous function. For brevity we will refer to this type of problem as a minimisation problem.

One approach in showing the existence of minimisers is by what are known as **direct methods**, that is we work with the functional $\mathcal{J}$ directly. It is known that in a finite-dimensional space any continuous function on a compact set attains its minimum; the basic idea in showing this is to find a minimising sequence belonging to a closed, bounded set and to extract a convergent subsequence and use continuity to prove that the limit is a minimiser. However, we are working in the Sobolev space $W^{1,1}(\Omega)$, and as this is an infinite-dimensional space it is not enough that the sequence belongs to a closed, bounded set, since closed, bounded sets are not necessarily compact in the norm topology. Instead we require the weaker notions of sequential weak compactness (Definition 3.1.8) and sequential weak lower semicontinuity (Definition 3.1.10 (ii)). Thus in order to show the existence of minimisers we require G to satisfy suitable coercivity, lower semicontinuity and convexity conditions. The coercivity conditions ensure that the minimising sequence (f_n) is bounded in $W^{1,p}(\Omega)$, $p > 1$, and thus contains a weakly convergent subsequence converging to some weak limit $\hat{f}$. The convexity and lower semicontinuity conditions then ensure that $\mathcal{J}(f)$ is sequentially weakly lower semicontinuous, by using results such as Theorem 3.1.13 or Theorem 3.1.21. Thus, in order to show existence, it remains to show that $\hat{f}$ is an admissible function (that is $\hat{f} \in \mathcal{A}$). It may well be the case that the minimisers are smoother functions than $W^{1,p}$ ($p > 1$), but this requires further analysis in each particular case.

3.2 Mathematical elasticity.

In this section we will introduce the terminology, notation and key results used in nonlinear elasticity. More detailed accounts of the development of the theory can be found in, for example, [25] and [51], as well as in some texts on continuum mechanics such as [36] and [65]. A brief account can be found in [35]. In [25] the approach is more analytic than applied, and a detailed description of the key results and ideas concerning the problem of minimising the total stored energy by variational techniques, as presented in [6], can also be found there.

Deformations.

Throughout we will let Ω be a bounded open subset of $\mathbb{R}^n$ $(n = 2, 3)$. We consider an elastic body occupying this region Ω in the absence of applied forces, and so we refer to Ω as the **reference configuration**. Upon applying these forces the body occupies a region $\boldsymbol{u}(\Omega)$ referred to as the **deformed configuration** and the map $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ is referred to as a **deformation**. We require $\boldsymbol{u}$ to be injective in order that interpenetration of the material does not occur. Let $\boldsymbol{x} := (x_1, \dots, x_n)^T$ be a point in $\Omega \subset \mathbb{R}^n$. Given a deformation

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{pmatrix} u^1(\boldsymbol{x}) \\ \vdots \\ u^m(\boldsymbol{x}) \end{pmatrix}$$

we define the Jacobian matrix

$$\nabla \boldsymbol{u} := \begin{pmatrix} u^1_{,1} & \cdots & u^1_{,n} \\ \vdots & & \vdots \\ u^m_{,1} & \cdots & u^m_{,n} \end{pmatrix} \text{ where } u^i_{,j} = \frac{\partial u^i}{\partial x_j}, \; i = 1, \dots, m, \; j = 1, \dots, n.$$

The matrix $\nabla \boldsymbol{u}$ is referred to as the **deformation gradient**. We require the deformation gradient to satisfy

$$\det(\nabla \boldsymbol{u}) > 0 \text{ a.e. in } \Omega. \tag{3.2.1}$$

This is an **orientation-preserving condition**. For $\boldsymbol{u} \in C^1(\Omega)$ this condition ensures local invertibility of $\boldsymbol{u}$ by the implicit function theorem. If the material is such that the only restriction on the deformations is (3.2.1) then the material is said to be **compressible**.

Throughout the rest of this section we will assume that $m = n$. We will also introduce the following notation: $M^{n \times n}$ is the set of all $n \times n$ matrices, $M^{n \times n}_+$ is the set of $n \times n$ matrices with positive determinant, $M^{n \times n}_1$ is the set of $n \times n$ matrices with determinant 1, $S^{n \times n}$ is the set of all $n \times n$ symmetric matrices, and $SO(n)$ is the set of all proper orthogonal $n \times n$ matrices.

The Cauchy stress tensor and the equilibrium equations.

We will assume throughout that we are considering a pure displacement boundary value problem, that is we define prescribed boundary conditions on the whole boundary. Now a body in the deformed configuration is subject to forces of two types:

1. applied body forces $\boldsymbol{b} : \boldsymbol{u}(\Omega) \to \mathbb{R}^n$ with $\boldsymbol{y} \mapsto \boldsymbol{b}(\boldsymbol{y})$ continuous; $\boldsymbol{b}(\boldsymbol{y})$ is a measure of the force per unit volume exerted at $\boldsymbol{y}$;

2. surface forces $\boldsymbol{t}$: {unit vectors} $\times\, \boldsymbol{u}(\Omega) \to \mathbb{R}^n$ with $\boldsymbol{y} \mapsto \boldsymbol{t}(\boldsymbol{n}, \boldsymbol{y})$ smooth; $\boldsymbol{t}(\boldsymbol{n}, \boldsymbol{y})$ is a measure of the force per unit area at $\boldsymbol{y}$ on any oriented surface with normal $\boldsymbol{n}$ through $\boldsymbol{y}$, and is sometimes referred to as the **Cauchy stress vector**.

The applied body forces $\boldsymbol{b}$ and the Cauchy stress vector $\boldsymbol{t}$ are assumed to be consistent with the **laws of force and moment balance**: for any subdomain D of $\boldsymbol{u}(\Omega)$ we have

$$\int_{\partial D} \boldsymbol{t}(\boldsymbol{n}(\boldsymbol{y}), \boldsymbol{y})\ \mathrm{dS} + \int_D \boldsymbol{b}(\boldsymbol{y})\ \mathrm{dy} = 0,$$

$$\int_{\partial D} \boldsymbol{y} \times \boldsymbol{t}(\boldsymbol{n}(\boldsymbol{y}), \boldsymbol{y})\ \mathrm{dS} + \int_D \boldsymbol{y} \times \boldsymbol{b}(\boldsymbol{y})\ \mathrm{dy} = 0$$

where $\boldsymbol{n}(\boldsymbol{y})$ is an outward unit normal to D at $\boldsymbol{y}$. In our work we will assume that $\boldsymbol{b} \equiv \mathbf{0}$. A consequence of these laws is **Cauchy's theorem**, which shows the existence of a smooth, symmetric, spatial tensor field $\boldsymbol{T}$ such that $\boldsymbol{t}(\boldsymbol{n}, \boldsymbol{y}) = \boldsymbol{T}(\boldsymbol{y})\boldsymbol{n}$, for every normal unit vector $\boldsymbol{n}$ and every $\boldsymbol{y} \in \boldsymbol{u}(\Omega)$. We call $\boldsymbol{T}$ the **Cauchy stress tensor**. Thus, the Cauchy stress tensor gives a measure of the surface force per unit area in the deformed configuration. (For details and a proof see, e.g., [36, Chapter V].)

As a consequence of Cauchy's theorem, in the absence of body forces, the **equilibrium equations** are of the form

$$\operatorname{div}(\boldsymbol{T}) = \mathbf{0} \tag{3.2.2}$$

and a deformation $\boldsymbol{u}$ satisfying (3.2.2) will be referred to as an **equilibrium solution**. We note that the **Piola-Kirchhoff stress tensor** $\boldsymbol{T}_{\mathrm{R}}$ is a measure of the force per unit area in the reference configuration and is often used, since in some problems involving solids it is not convenient to work directly with $\boldsymbol{T}$ as this requires knowledge of the deformed configuration in advance. The relation between the two is that

$$\boldsymbol{T}(\boldsymbol{u}) = (\det(\nabla \boldsymbol{u}(\boldsymbol{x})))^{-1} \boldsymbol{T}_{\mathrm{R}}(\boldsymbol{x}) (\nabla \boldsymbol{u}(\boldsymbol{x}))^T.$$

Hyperelastic materials.

A material is called **hyperelastic** if there exists a function $W : \Omega \times M_+^{n\times n} \to \mathbb{R}$ such that for each $\boldsymbol{u}$ the total energy stored in the body is

$$E(\boldsymbol{u}) = \int_\Omega W(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}))\ \mathrm{dx}. \tag{3.2.3}$$

The function $W(\boldsymbol{x}, F)$ is commonly referred to as the **stored energy function**. If W is independent of $\boldsymbol{x}$ then we say that the material is **homogeneous**. For a compressible material the relation between the stored energy function and the Piola-Kirchhoff stress tensor is

$$\boldsymbol{T}_{\mathrm{R}}(\boldsymbol{x}) = \frac{\partial W}{\partial F}(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x})).$$

Also the equilibrium equations in the case of a compressible hyperelastic material (under zero body forces) can be written as

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial W}{\partial F^i_\alpha}(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}))\right) = 0, \qquad i = 1, \dots, n, \tag{3.2.4}$$

which are the Euler-Lagrange equations for (3.2.3).
We say that the stored energy function of a hyperelastic material is **frame-indifferent** if and only if at all points $\boldsymbol{x} \in \Omega$, $W(\boldsymbol{x}, QF) = W(\boldsymbol{x}, F)$ for all $F \in M_+^{n\times n}$ and $Q \in SO(n)$ (this corresponds to invariance under a change of observer). We say that the stored energy function of a hyperelastic material is **isotropic** at $\boldsymbol{x} \in \Omega$ if and only if $W(\boldsymbol{x}, FQ) = W(\boldsymbol{x}, F)$ for all $F \in M_+^{n\times n}$ and $Q \in SO(n)$ (this corresponds to the response of the material having no preferred direction). We assume that our stored energy function is both frame-indifferent and isotropic.

Incompressible materials.

We say that a material is **incompressible** if it only admits deformations $\boldsymbol{u}$ satisfying the constraint

$$\det(\nabla \boldsymbol{u}) = 1 \text{ a.e. in } \Omega. \tag{3.2.5}$$

This is an example of an **internal constraint**, and any deformations satisfying this particular constraint must be locally volume preserving. For an incompressible hyperelastic material the total energy stored in the body is still of the form

(3.2.3) but the stored energy function W is only defined for those deformations $\boldsymbol{u}$ satisfying (3.2.5) (and thus $W : \Omega \times M_1^{n\times n} \to \mathbb{R}$). The Piola-Kirchhoff stress tensor for an incompressible hyperelastic material is now given by

$$\boldsymbol{T}_{\mathrm{R}}(\boldsymbol{x}) = -\bar{P}(\boldsymbol{x})(\nabla \boldsymbol{u}(\boldsymbol{x}))^T + \frac{\partial W}{\partial F}(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x}))$$

where $\bar{P}(\boldsymbol{x})$ is a Lagrange multiplier due to (3.2.5) and is often referred to as a **hydrostatic pressure**. Hence, the equilibrium equations are

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\mathrm{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))^\alpha_i\right] = 0, \; i = 1, \ldots, n, \tag{3.2.6}$$

where $\mathrm{adj}(F)$ is the adjugate matrix of F, which is the transpose of the matrix of cofactors of F. (For details see, for example, [51] pp 198-201, [58] or [69] pp 71-72.) We note that (3.2.6) are the Euler-Lagrange equations for

$$E(\boldsymbol{u}) = \int_\Omega W(\boldsymbol{x}, \nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})\left\{\det(\nabla \boldsymbol{u}(\boldsymbol{x})) - 1\right\} \, \mathrm{dx}. \tag{3.2.7}$$

We also note that for $\boldsymbol{u} \in C^2(\Omega)$ satisfying (3.2.6) $\bar{P}(\boldsymbol{x})$ is unique up to a constant. To show this, we use an argument from the proof of Theorem 4.3 of [8]. If $\bar{P}_1(\boldsymbol{x})$ and $\bar{P}_2(\boldsymbol{x})$ are two possible pressure terms and $\bar{Q}(\boldsymbol{x}) = \bar{P}_1(\boldsymbol{x}) - \bar{P}_2(\boldsymbol{x})$ then

$$\int_\Omega \bar{Q}(\boldsymbol{x})(\mathrm{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))^T . \nabla \boldsymbol{\phi}(\boldsymbol{x}) \, \mathrm{dx} = 0$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$. Now $\boldsymbol{u}$ is a C^2 measure preserving diffeomorphism from Ω onto $\boldsymbol{u}(\Omega)$. Hence, in index notation

$$\int_{\boldsymbol{u}(\Omega)} \bar{Q}(\boldsymbol{x}(\boldsymbol{u}))\frac{\partial \phi^i(\boldsymbol{u})}{\partial u^i} \, \mathrm{du} = 0$$

for all $\boldsymbol{\phi} \in C_0^\infty(\boldsymbol{u}(\Omega))$, which implies that $\bar{Q}$ is constant.

A note on convexity conditions.

A key condition in showing the existence of minimisers in one-dimensional elasticity ($n = 1$) is that the stored energy function W is convex in its dependence on $\nabla \boldsymbol{u}$. As stated in §3.1 convexity, along with lower semicontinuity and coercivity of a function E, is sufficient to ensure that E attained a minimum. However, in higher dimensional problems ($n > 1$), as indicated in [7], this condition is unrealistic to apply as a general constitutive hypothesis, as convexity of W conflicts with the requirement that the stored energy function is frame-indifferent, and strict convexity rules out multiple solutions that can occur, for example in buckling. A more realistic assumption to make is that W is **quasiconvex**. This concept was introduced by Morrey [48]. We say that W is **quasiconvex at a point** $(\boldsymbol{x}, F) \in \Omega \times M_{+}^{n\times n}$ if

$$\int_D W(\boldsymbol{x}, F + \nabla \boldsymbol{v}(\boldsymbol{y}))\, \mathrm{dy} \geq \mathrm{meas}(D) \times W(\boldsymbol{x}, F)$$

for every bounded open subset $D \subset \mathbb{R}^n$ and for every $\boldsymbol{v} \in W_0^{1,\infty}(D)$ satisfying $F + \nabla \boldsymbol{v} \in M_{+}^{n\times n}$ for all $\boldsymbol{y} \in \Omega$. W is **quasiconvex** if it is quasiconvex at every $(\boldsymbol{x}, F) \in \Omega \times M_{+}^{n\times n}$.

Under suitable growth and continuity conditions quasiconvexity is a necessary and sufficient condition for weak lower semicontinuity. For details, see [48, Theorem 4.5.5] page 117 or [1] under weaker growth hypotheses. However, since quasiconvexity is not a pointwise condition, it is not easy to verify. Also, the growth hypotheses assumed in both of these results are too stringent to apply to elasticity since they prohibit singular behaviour of the stored energy function.

A weaker condition than quasiconvexity is that of **rank-one convexity**. This condition is more straightforward to verify. W is **rank-one convex** if

$$W(\boldsymbol{x}, F + (1-\lambda)\boldsymbol{a} \otimes \boldsymbol{b}) \leq \lambda W(\boldsymbol{x}, F) + (1-\lambda) W(\boldsymbol{x}, F + \boldsymbol{a} \otimes \boldsymbol{b})$$

for all $F \in M^{n\times n}$, $\lambda \in [0,1]$, $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^n$ with $F + \lambda(\boldsymbol{a} \otimes \boldsymbol{b}) \in M^{n\times n}$.

A stronger condition than quasiconvexity is that of **polyconvexity**. This was introduced by Ball (cf [6], [7]). The stored energy function W is **polyconvex** if:

1. $(n = 2)$ there exists a function $g : \Omega \times M^{2\times 2} \times (0, \infty) \to \mathbb{R}$ with $g(\boldsymbol{x}, \cdot, \cdot)$ convex for each $\boldsymbol{x} \in \Omega$ such that $W(\boldsymbol{x}, F) = g(\boldsymbol{x}, F, \det(F))$ for all $\boldsymbol{x} \in \Omega$ and $F \in M^{2\times 2}$ with $\det(F) > 0$;

2. $(n = 3)$ there exists a function $g : \Omega \times M^{3\times 3} \times M^{3\times 3} \times (0, \infty) \to \mathbb{R}$ with $g(\boldsymbol{x}, \cdot, \cdot, \cdot)$ convex for each $\boldsymbol{x} \in \Omega$ such that

$$W(\boldsymbol{x}, F) = g(\boldsymbol{x}, F, \mathrm{adj}(F), \det(F))$$

for all $\boldsymbol{x} \in \Omega$ and $F \in M^{3\times 3}$ with $\det(F) > 0$.

Examples of polyconvex stored energy functions are $W(F) = \frac{1}{2}|F|^2 + h(\det(F))$ where h is a convex function.

The four convexity conditions are related in the following way:

$$W \text{ convex } \Rightarrow W \text{ polyconvex } \Rightarrow W \text{ quasiconvex } \Rightarrow W \text{ rank-one convex.}$$

All of these implications are one-way only. $W(\boldsymbol{x}, F) = \det(F)$ is a simple example of a function that is polyconvex but not convex. It has been known for some time that quasiconvexity does not imply polyconvexity, and examples can be found for all cases $n \geq 2$ (see, e.g., [73]), although Ball points out in [6] that there are few known examples of quasiconvex functions that are not polyconvex and applicable to elasticity. An example of a function that is rank-one convex but not quasiconvex is given in [66], for the case $n \geq 3$. The case $n = 2$ is still open, and has recently been investigated (see, e.g., [54] or [55] - these seem to make the suggestion that rank-one convexity is equivalent to quasiconvexity for the case $n = 2$). A further discussion on special conditions under which equivalence of these convexity conditions can be established can be found in [27, Chapter 4]. With regards to quasiconvexity, we briefly consider the functional

$$E(\boldsymbol{u}) = \int_{\Omega} W(\nabla \boldsymbol{u}) \, \mathrm{dx}$$

defined on $\mathcal{A} = \{\boldsymbol{u} \in W^{1,2}(\Omega) : \boldsymbol{u} = \bar{\boldsymbol{u}} \text{ on } \partial\Omega\}$. Following [29] we say that W is **uniformly strictly quasiconvex** provided

$$\int_{\Omega} W(F) + \gamma |\nabla \boldsymbol{v}|^2 \, \mathrm{dx} \leq \int_{\Omega} W(F + \nabla \boldsymbol{v}) \, \mathrm{dx} \tag{3.2.8}$$

for some constant $\gamma > 0$ and all $F \in M^{m \times n}$, $\boldsymbol{v} \in C_0^{\infty}(\Omega)$. One consequence of this is the following result due to Evans.

Proposition 3.2.1 ([29], page 27) *Suppose W is uniformly strictly quasiconvex, and suppose that $W(F)$ is such that $W(F) \leq C(1 + |F|^2)$. If $\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}$ in $W^{1,2}(\Omega)$ and $E(\boldsymbol{u}_n) \to E(\boldsymbol{u})$, then $\boldsymbol{u}_n \to \boldsymbol{u}$ in $W^{1,2}(\Omega)$.*

Throughout the remainder of this work, we will work with stored energy functions that are polyconvex.

The problem of showing existence of energy minimisers in nonlinear elasticity.

The basic problem in nonlinear elasticity is to determine the existence (or otherwise) of deformations $\boldsymbol{u} : \Omega \to \mathbb{R}^n$ that minimise the total energy E among those deformations that are one-to-one almost everywhere and, in the case of the pure displacement boundary value problem, are equal to some given function $\boldsymbol{u}_0$ on the boundary $\partial\Omega$. In the case of a hyperelastic material the energy is given by (3.2.3) and we will assume

(i) that the stored energy function $W(\boldsymbol{x}, F)$ is **not necessarily convex** with respect to F,

(ii) that $W(\boldsymbol{x}, F) \to \infty$ as $\det(F) \to 0, \infty$.

We briefly mention that there is an approach in showing the existence of minimisers to the boundary value problem that is based on the implicit function theorem. This method was developed by Stoppelli and is discussed and extended in detail by Valent in [70]. (A brief discussion appears in [25, Chapter 6].) The basic idea is to start from one known solution and use the implicit function theorem to obtain solutions in a neighbourhood of the known solution. Valent [70] shows this to be the case for nonlinear boundary value problems. However, this method is not applicable to the nonlinear problems we are considering since our problem is to minimise functionals that are coercive over nonconvex subsets of Sobolev spaces, where W must satisfy (i) and (ii).
The method that is applicable and takes these conditions into account was devised by Ball. For full details see [6]. (This is also discussed in [25, Chapter 7].) The method he developed considers a minimising sequence of functions $(\boldsymbol{u}_n)$ on a appropriate set of admissible deformations $\mathcal{A}$ in $W^{1,p}(\Omega)$ satisfying (3.2.1) and prescribed boundary conditions. Ball shows that this sequence is bounded as a consequence of coercivity, extracts a weakly convergent subsequence $(\boldsymbol{u}_j)$ converging weakly to some limit $\bar{\boldsymbol{u}}$, shows that $\bar{\boldsymbol{u}} \in \mathcal{A}$ and that

$$\int_\Omega W(\boldsymbol{x}, \nabla\bar{\boldsymbol{u}}(\boldsymbol{x}))\, \mathrm{dx} \leq \liminf_{j\to\infty} \int_\Omega W(\boldsymbol{x}, \nabla\boldsymbol{u}_j(\boldsymbol{x}))\, \mathrm{dx} \tag{3.2.9}$$

in order to show the existence of minimisers. He achieved this by replacing convexity with the weaker assumptions of polyconvexity and (for the case $n = 3$) and by assuming that the stored energy function satisfies the coercivity condition

$$W(\boldsymbol{x}, F) \geq C_1(|F|^p + |\mathrm{adj}(F)|^q + (\det(F))^r) + C_2 \tag{3.2.10}$$

where

$$p \geq 2,\ q \geq \frac{p}{p-1} \text{ and } r > 1. \tag{3.2.11}$$

Also, the set of admissible deformations is of the form

$$\mathcal{A} = \{\boldsymbol{u} \in W^{1,p}(\Omega) : \operatorname{adj}(\nabla \boldsymbol{u}) \in L^q(\Omega), \det(\nabla \boldsymbol{u}) \in L^r(\Omega), \\ \boldsymbol{u} = \boldsymbol{u}_0 \text{ a.e. on } \partial\Omega, \det(\nabla \boldsymbol{u}) > 0 \text{ a.e. in } \Omega\}$$

where p, q and r satisfy (3.2.11).
As W is polyconvex it can be treated as a convex function of nineteen independent variables if $n = 3$ (or five if $n = 2$). Ball showed that the sequence

$$((\boldsymbol{u}_n, \operatorname{adj}(\nabla \boldsymbol{u}_n), \det(\nabla \boldsymbol{u}_n))) \subset W^{1,p}(\Omega) \times L^q(\Omega) \times L^r(\Omega),$$

where p, q and r satisfy (3.2.11), is bounded. Thus by Theorem 3.1.7 there exists a weakly convergent subsequence $((\boldsymbol{u}_j, \operatorname{adj}(\nabla \boldsymbol{u}_j), \det(\nabla \boldsymbol{u}_j)))$ converging weakly to $(\bar{\boldsymbol{u}}, H, d)$ in $W^{1,p}(\Omega) \times L^q(\Omega) \times L^r(\Omega)$, where p, q and r satisfy (3.2.11), and Ball showed that in fact $H = \operatorname{adj}(\nabla \bar{\boldsymbol{u}})$ and $d = \det(\nabla \bar{\boldsymbol{u}})$. From here sequential weak lower semicontinuity of E can be shown. (Also in [6] are existence results for mixed displacement-traction problems.)
Throughout the rest of this work we assume that the material is homogeneous. We now give an existence result due to Ball & Murat (cf [13]). This result is a refinement of the existence results found in [6] as the growth condition on W is independent of $\det(F)$. We will consider a pure displacement problem such that the deformation must satisfy $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$ on $\partial\Omega$ (although the result in [13] also holds for mixed displacement-zero traction boundary value problems). For a body of homogeneous material, the energy functional is given by

$$E(\boldsymbol{u}) = \int_\Omega W(\nabla \boldsymbol{u}) \, \mathrm{dx}$$

and we seek to minimise on the set

$$\mathcal{A} = \{\boldsymbol{u} \in W^{1,p}(\Omega) : \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}) \text{ on } \partial\Omega \text{ and } E(\boldsymbol{u}) < \infty\}.$$

This is achieved by the following result. We will present the result for the case $n = 3$ as given in [13].

Theorem 3.2.2 ([13, Theorem 6.1]) *Let W satisfy the following hypotheses:*

(A) *W is continuous in F if $\det(F) > 0$, $W(F) = \infty$ if and only if $\det(F) \leq 0$ and $W(F) \to \infty$ as $\det(F) \to 0_+$,*

(B) *W is polyconvex, that is there exists a convex function $g : M^{3\times3} \times M^{3\times3} \times (0,\infty) \to \mathbb{R}$ such that $W(F) = g(F, \operatorname{adj}(F), \det(F))$ for all $F \in M_+^{3\times3}$,*

(C) *$W(F) \geq C + K(|F|^p + |\operatorname{adj}(F)|^q)$ for all $F \in M_+^{3\times3}$ where $C > 0$, K are constants, $p \geq 2$ and $q \geq \dfrac{p}{p-1}$.*

Let $\mathcal{A}$ be nonempty. Then E attains an absolute minimum on $\mathcal{A}$ and the minimiser satisfies $\det(\nabla \boldsymbol{u}(\boldsymbol{x})) > 0$ a.e. on Ω.

Remark 3.2.3 *We note that for $n = 2$ hypotheses (B) and (C) can be replaced by*

(B$'$) *W is polyconvex, that is there exists a convex function $g : M^{2\times2} \times (0,\infty) \to \mathbb{R}$ such that $W(F) = g(F, \det(F))$ for all $F \in M_+^{2\times2}$,*

(C$'$) *$W(F) \geq C + K|F|^p$ for all $F \in M_+^{2\times2}$ where $C > 0$, K are constants, $p \geq 2$,*

and Theorem 3.2.2 remains valid. (See the note following [13, Theorem 6.1].)

Chapter 4

Rotationally symmetric deformations of a two-dimensional annulus.

The work presented in the rest of this thesis is motivated by the paper of Post & Sivaloganathan [57], in particular the example of §4. There they consider a nonlinear elastic body which occupies the annular region

$$A = \{\boldsymbol{x} \in \mathbb{R}^2; a < |\boldsymbol{x}| < b\} \tag{4.0.1}$$

in its reference state, and deformations $\boldsymbol{u} : A \to \mathbb{R}^2$ satisfying the boundary condition $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial A$. They require $\boldsymbol{u}$ to satisfy the local invertibility condition

$$\det(\nabla \boldsymbol{u}) > 0 \text{ a.e. in } A. \tag{4.0.2}$$

They assume that the material is hyperelastic (under zero body forces) and homogeneous, so that the stored energy E associated with the deformation $\boldsymbol{u}$ is given by

$$E(\boldsymbol{u}) = \int_A W(\nabla \boldsymbol{u}(\boldsymbol{x}))\ \mathrm{dx}, \tag{4.0.3}$$

where $W : M^{2\times 2} \to \mathbb{R}$ is the stored energy function. They require W to satisfy hypotheses (A), (B′) and (C′) as given in Theorem 3.2.2 and Remark 3.2.3, and

$$W(F) \to \infty \text{ as } \det(F) \to 0,\ \infty. \tag{4.0.4}$$

In [57] it is shown that under these conditions multiple equilibria exist and that they are local minimisers (see [57, Theorem 3.1 and Remark 3.2]). Now the example of [57, §4] restricts attention to rotationally symmetric deformations $\boldsymbol{u} : \bar{A} \to \bar{A}$ of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \rho(R) \begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix}, \tag{4.0.5}$$

where $R = |\boldsymbol{x}| \in [a, b]$, $\theta \in [0, 2\pi)$, and $\rho : [a, b] \to [a, b]$, $\psi : [a, b] \to \mathbb{R}$, and polyconvex stored energy functions of the form

$$W(F) = \frac{1}{2}|F|^2 + h(\det(F)) \tag{4.0.6}$$

for all $F \in M^{2\times 2}$, where $h : (0, \infty) \to \mathbb{R}$ is convex, C^2 and $h(d) \to \infty$ as $d \to 0, \infty$. Note that W is both frame-indifferent and isotropic. As noted in [57] we can write

$$\begin{aligned} \nabla \boldsymbol{u} = &\begin{pmatrix} \rho' \cos(\theta + \psi) - \rho\psi' \sin(\theta + \psi) \\ \rho' \sin(\theta + \psi) + \rho\psi' \cos(\theta + \psi) \end{pmatrix} \otimes \frac{\boldsymbol{x}}{R} \\ &\qquad + \frac{1}{R} \begin{pmatrix} -\rho \sin(\theta + \psi) \\ \rho \cos(\theta + \psi) \end{pmatrix} \otimes \frac{1}{R} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \end{aligned} \tag{4.0.7}$$

Also

$$|\nabla \boldsymbol{u}|^2 = (\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2 \text{ and } \det(\nabla \boldsymbol{u}) = \rho' \frac{\rho}{R}. \tag{4.0.8}$$

Thus, the total stored energy $E(\boldsymbol{u})$ corresponding to such a deformation is given by

$$E(\boldsymbol{u}) = 2\pi \hat{I}(\rho, \psi) = 2\pi \int_a^b R \left[\frac{1}{2} \left\{ (\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2 \right\} + h\left(\rho' \frac{\rho}{R}\right) \right] dR, \tag{4.0.9}$$

and it is shown in [57] that for each $N \in \mathbb{N} \cup \{0\}$ a minimiser exists for $\hat{I}$ on the set

$$\begin{aligned} \mathcal{A}_N^{\text{sym}} = \{(\rho, \psi) \in W^{1,1}((a, b)) : \rho(a) = a, \rho(b) = b, \rho'(R) > 0 \text{ a.e. on } (a, b), \\ \psi(a) = 0, \psi(b) = 2N\pi\}. \end{aligned} \tag{4.0.10}$$

They state that a minimiser (ρ, ψ) is in $C^2((a,b))$, satisfies the Euler-Lagrange equations for $\hat{I}(\rho, \psi)$ and that a smooth solution of those equations gives rise to a corresponding solution $\boldsymbol{u}$ of the full Euler-Lagrange equations for E, that is

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u}(\boldsymbol{x}))\right] = 0, \;\; \boldsymbol{x} \in A, \;\; i = 1, 2. \tag{4.0.11}$$

(See [57, Remark 4.3].) It is also shown that the angle of twist ψ for a symmetric minimiser is monotonic in R.

In this chapter we will discuss two modifications to the example of [57, §4]. In §4.1 we will extend the results of [57, §4] by proving the existence of minimisers for a more general class of polyconvex stored energy functions. We will also give a proof showing that the minimisers in this general case satisfy the Euler-Lagrange equations in the symmetric variables, and we show that the rotationally symmetric minimisers also give rise to a solution of the full two-dimensional Euler-Lagrange equations. In §4.2 we will consider a model problem in which we put $h \equiv 0$ into (4.0.6). In this case the formal Euler-Lagrange equations are the vector Laplace equations and can be solved explicitly, and an explicit form of a potential minimiser is constructed. Since $W(F) \not\to \infty$ as $\det(F) \to 0$ in this case there is the possibility that minimising sequences and potential minimisers may be *degenerate* in that $\det(\nabla \boldsymbol{u}) = 0$ on a set of non-zero measure. It will be shown in Chapter 6 that this construction minimises the energy functional in a more general class of deformations.

4.1 An existence result for a general class of stored energy functions.

As in [57, §4] we restrict our attention to deformations $\boldsymbol{u}$ as given by (4.0.5), but we consider a more general class of polyconvex stored energy functions W of the form

$$W(\nabla \boldsymbol{u}) = \tilde{g}(|\nabla \boldsymbol{u}|, \det(\nabla \boldsymbol{u})), \tag{4.1.1}$$

where $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ is convex. Due to (4.0.8), we can write $W(\nabla \boldsymbol{u})$ in the form

$$W(\nabla \boldsymbol{u}) = \tilde{g}\left(\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}, \rho'\frac{\rho}{R}\right), \tag{4.1.2}$$

and hence the total stored energy corresponding to a rotationally symmetric deformation of the form (4.0.5) is

$$E(\boldsymbol{u}) = 2\pi I(\rho, \psi) = 2\pi \int_a^b R\tilde{g}\left(\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}, \rho'\frac{\rho}{R}\right) dR. \quad (4.1.3)$$

In order to show the existence of minimisers we require the following preliminary result.

Lemma 4.1.1 *Let $\tilde{g}$ be of the form given by (4.1.2), and let $\tilde{g}(\cdot,\cdot)$ be convex and $\tilde{g}_{,1} > 0$. Then $\tilde{g}$ is convex in (ρ', ψ').*

Proof: For sufficiently smooth deformations $\boldsymbol{u}$ of the type (4.0.5) we have $\nabla \boldsymbol{u}$ of the form (4.0.7). Now, put $\rho' = \lambda\kappa + (1-\lambda)\xi$ and $\psi' = \lambda\varphi + (1-\lambda)\chi$. Then we can write

$$\begin{aligned} A = \lambda\Bigg\{ & \begin{pmatrix} \kappa\cos(\theta+\psi) - \rho\varphi\sin(\theta+\psi) \\ \kappa\sin(\theta+\psi) + \rho\varphi\cos(\theta+\psi) \end{pmatrix} \otimes \frac{\boldsymbol{x}}{R} \\ & + \frac{1}{R}\begin{pmatrix} -\rho\sin(\theta+\psi) \\ \rho\cos(\theta+\psi) \end{pmatrix} \otimes \frac{1}{R}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\Bigg\} \\ +(1-\lambda)\Bigg\{ & \begin{pmatrix} \xi\cos(\theta+\psi) - \rho\chi\sin(\theta+\psi) \\ \xi\sin(\theta+\psi) + \rho\chi\cos(\theta+\psi) \end{pmatrix} \otimes \frac{\boldsymbol{x}}{R} \\ & + \frac{1}{R}\begin{pmatrix} -\rho\sin(\theta+\psi) \\ \rho\cos(\theta+\psi) \end{pmatrix} \otimes \frac{1}{R}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}\Bigg\}. \end{aligned}$$

Thus $A = \lambda F_1 + (1-\lambda)F_2$, say, and hence $|A| = |\lambda F_1 + (1-\lambda)F_2| \leq \lambda|F_1| + (1-\lambda)|F_2|$ by the triangle inequality. Evaluating $|A|$, $|F_1|$ and $|F_2|$ gives us that

$$\begin{aligned} &\left|(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right|^{\frac{1}{2}} \\ &\qquad \leq \lambda\left|\kappa^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\varphi)^2\right|^{\frac{1}{2}} + (1-\lambda)\left|\xi^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\chi)^2\right|^{\frac{1}{2}}. \end{aligned}$$

Now, as $\tilde{g}_{,1} > 0$, we have that

$$\begin{aligned} \tilde{g}\left(\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}, \rho'\frac{\rho}{R}\right) \leq \tilde{g}\Bigg(\lambda&\left|\kappa^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\varphi)^2\right|^{\frac{1}{2}} \\ + (1-\lambda)\left|\xi^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\chi)^2\right|^{\frac{1}{2}}&, (\lambda\kappa + (1-\lambda)\xi)\frac{\rho}{R}\Bigg), \end{aligned} \quad (4.1.4)$$

and, as $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ is convex, it follows that

$$\tilde{g}\left(\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}, \rho'\frac{\rho}{R}\right) \leq$$
$$\lambda\tilde{g}\left(\left|\kappa^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\varphi)^2\right|^{\frac{1}{2}}, \kappa\frac{\rho}{R}\right) + (1-\lambda)\tilde{g}\left(\left|\xi^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\chi)^2\right|^{\frac{1}{2}}, \xi\frac{\rho}{R}\right),$$

and so $\tilde{g}$ is convex in (ρ', ψ'). $\square$

We now prove the generalisation of the result in [57, §4].

Proposition 4.1.2 *Let W be of the form*

$$W(F) = \tilde{g}(|F|, \det(F)) \tag{4.1.5}$$

where $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ is C^2, convex and such that $\tilde{g}_{,1} > 0$ for all F, $\tilde{g}(|F|, \det(F)) \to \infty$ as $\det(F) \to 0$, ∞ and

$$\tilde{g}(|F|, \det(F)) > C_1|F|^p + C_2 \tag{4.1.6}$$

where $C_1 > 0$ and C_2 are constants, $p \geq 2$. Then there exists a minimiser for $I(\rho, \psi)$ on $\mathcal{A}_N^{\text{sym}}$ for each $N \in \mathbb{N} \cup \{0\}$, where $I(\rho, \psi)$ is given by (4.1.3) and $\mathcal{A}_N^{\text{sym}}$ is given by (4.0.10).

Proof: The proof of this uses the direct method of the calculus of variations. Let $(\rho_n, \psi_n) \in \mathcal{A}_N^{\text{sym}}$ be a minimising sequence for I on $\mathcal{A}_N^{\text{sym}}$, that is

$$\begin{aligned} I(\rho_n, \psi_n) &= \int_a^b R\tilde{g}\left(\left\{(\rho_n')^2 + \left(\frac{\rho_n}{R}\right)^2 + (\rho_n\psi_n')^2\right\}^{\frac{1}{2}}, \rho_n'\frac{\rho_n}{R}\right) dR \\ &\to \inf_{\mathcal{A}_N^{\text{sym}}} I \quad \text{as } n \to \infty. \end{aligned}$$

It follows from (4.0.8) and (4.1.6) that

$$W(\nabla \boldsymbol{u}) \geq C_1\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{p}{2}} + C_2,$$

where C_1 and C_2 are constants, and so

$$I(\rho_n, \psi_n) \geq aC_1\int_a^b \left\{(\rho_n')^2 + \left(\frac{\rho_n}{R}\right)^2 + (\rho_n\psi_n')^2\right\}^{\frac{p}{2}} dR + aC_2(b-a).$$

Also, by Hölder's inequality

$$\left\{\int_a^b (\rho_n')^2 + \left(\frac{\rho_n}{R}\right)^2 + (\rho_n\psi_n')^2 \, dR\right\}^{\frac{p}{2}}$$
$$\leq \left(\int_a^b \left\{(\rho_n')^2 + \left(\frac{\rho_n}{R}\right)^2 + (\rho_n\psi_n')^2\right\}^{\frac{p}{2}} dR\right)\left(\int_a^b 1^{\frac{p}{p-2}} dR\right)^{\frac{p}{2}-1}.$$

Hence we now have that

$$I(\rho_n, \psi_n) \geq C_1 \left\{\frac{1}{b^2}\int_a^b (\rho_n)^2 dR + \int_a^b (\rho_n')^2 dR + a^2 \int_a^b (\psi_n')^2 dR\right\}^{\frac{p}{2}} + C_2$$

for all $n \in \mathbb{N}$. Since $\psi(a) = 0$ we can say

$$a^2 \int_a^b (\psi_n')^2 dR \geq \frac{a^2}{2}\int_a^b (\psi_n')^2 dR + \frac{a^2}{2(b-a)^2}\int_a^b (\psi_n)^2 dR,$$

due to Poincaré's inequality (see, e.g., [3, page 4]), and from this it follows that

$$I(\rho_n, \psi_n) \geq C_1 \left\{D_1 \parallel \rho_n \parallel_{1,2}^2 + D_2 \parallel \psi_n \parallel_{1,2}^2\right\}^{\frac{p}{2}} + C_2,$$

where D_1, D_2 are positive constants. Hence, $I(\rho, \psi)$ is coercive in $W^{1,2}((a,b))$, and hence $((\rho_n, \psi_n))_{n=1}^{\infty}$ is a bounded sequence in $W^{1,2}((a,b))$. Hence it has a weakly convergent subsequence still labelled (ρ_n, ψ_n) converging weakly to (ρ, ψ) in $W^{1,2}((a,b))$, where (ρ, ψ) satisfies the boundary conditions. We assume that $\tilde{g}$ is continuous, as W is continuous. Now, for $\boldsymbol{u}$ of the form (4.0.5) we define

$$\Phi(R, (\rho, \psi), (\rho', \psi')) := R\tilde{g}(|\nabla \boldsymbol{u}|, \det(\nabla \boldsymbol{u})).$$

It can be shown that Φ is continuous in R and that $\Phi(R, ., .)$ is continuous in (ρ, ψ) and (ρ', ψ') for all R in $[a, b]$. As $\tilde{g}$ is such that $\tilde{g}_{,1} > 0$ and $\tilde{g}(\cdot, \cdot)$ is convex, then by Lemma 4.1.1, $\Phi(R, (\rho, \psi), \cdot)$ is convex. Hence, using Theorem 3.1.21 with $y_n = (\rho_n, \psi_n)$, $z_n = (\rho_n', \psi_n')$, $y = (\rho, \psi)$ and $z = (\rho', \psi')$, it follows that I is sequentially weakly lower semicontinuous on (ρ_n, ψ_n) and it follows that

$$\inf_{\mathcal{A}_N^{\mathrm{sym}}} I = \liminf_{n \to \infty} I(\rho_n, \psi_n) \geq I(\rho, \psi)$$

so that in particular $I(\rho,\psi) < \infty$. By the properties of $\tilde{g}$, it follows that $\rho'(R) > 0$ for a.e. $R \in (a,b)$ and so $(\rho,\psi) \in \mathcal{A}_N^{\mathrm{sym}}$. Hence

$$I(\rho,\psi) = \inf_{\mathcal{A}_N^{\mathrm{sym}}} I,$$

and thus there exists a minimiser for $I(\rho,\psi)$ on $\mathcal{A}_N^{\mathrm{sym}}$. $\square$

We next define

$$\Psi(t_1,t_2,t_3) := \tilde{g}\left(\left\{(t_1)^2 + (t_2)^2 + (t_2 t_3)^2\right\}^{\frac{1}{2}}, t_1 t_2\right) \tag{4.1.7}$$

where t_1 corresponds to ρ', t_2 to $\dfrac{\rho}{R}$ and t_3 to ψ'. With this, we now show that any minimiser is C^2 and satisfies the Euler-Lagrange equations in terms of ρ and ψ.

Proposition 4.1.3 *Let $W(F) = \tilde{g}(|F|, \det(F))$ be such that $\tilde{g}$ satisfies the same hypotheses as for Proposition 4.1.2, and in addition that $\tilde{g}(|F|, \det(F))$ is strictly convex in (ρ', ψ') and satisfies the following:*

(R1) *If S is bounded, then $\tilde{g}_{,2}(S,T) \to -\infty$ as $T \to 0$ uniformly in S in some compact set;*

(R2) *$\tilde{g}_{,1}(S,T)$ is bounded if and only if S and T are bounded.*

Suppose also that there exist $\delta > 0$, M such that

$$|\Psi_{,2}(t_1, \alpha t_2, t_3) t_2| < M(\Psi(t_1,t_2,t_3) + 1) \qquad \text{if } |\alpha - 1| < \delta, \tag{4.1.8}$$

where Ψ is given by (4.1.7). If $(\hat{\rho}, \hat{\psi})$ is a minimiser of I on $\mathcal{A}_N^{\mathrm{sym}}$, then $(\hat{\rho}, \hat{\psi}) \in C^2((a,b))$ and $(\hat{\rho}, \hat{\psi})$ satisfies the Euler-Lagrange equations

$$\frac{d}{dR}\left[R\left\{\frac{\rho'}{S}\tilde{g}_{,1} + \frac{\rho}{R}\tilde{g}_{,2}\right\}\right] - \frac{1}{S}\left\{\frac{\rho}{R} + R\rho(\psi')^2\right\}\tilde{g}_{,1} - \rho'\tilde{g}_{,2} = 0 \tag{4.1.9}$$

and

$$\frac{d}{dR}\left[\frac{R\rho^2\psi'}{S}\tilde{g}_{,1}\right] = 0 \tag{4.1.10}$$

for all R in $[a,b]$, where $S = |\nabla \boldsymbol{u}| = \left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}$.

Proof: The proof follows a similar pattern to the proof of [60, Proposition 0.3] (see [60, §7]) and is in four steps. We first show that $\hat{\rho}$ and $\hat{\psi}$ satisfy the weak form of the Euler-Lagrange equations (Steps 1 & 2), then show that $(\hat{\rho},\hat{\psi}) \in C^1((a,b))$ (Step 3) and from there that $(\hat{\rho},\hat{\psi}) \in C^2((a,b))$, so that $\hat{\rho}$ and $\hat{\psi}$ satisfy the Euler-Lagrange equations (Step 4).

Step 1. Let $k \in (1,\infty)$, $l \in (0,\infty)$ and define $\mathcal{S}_k$, $\mathcal{S}_l$ by

$$\mathcal{S}_k := \left\{R \in (a,b)\,;\,\frac{1}{k} < \hat{\rho}'(R) < k\right\} \text{ and } \mathcal{S}_l := \left\{R \in (a,b)\,;\,\left|\hat{\psi}'(R)\right| < l\right\}.$$

Let $v \in L^\infty(a,b)$ satisfy

$$\int_{\mathcal{S}_k} v(R)dR = 0, \tag{4.1.11}$$

and put

$$\hat{\rho}_\varepsilon(R) = \hat{\rho}(R) + \varepsilon \int_a^R v(\tau)\chi_k(\tau)d\tau, \tag{4.1.12}$$

where χ_k is the characteristic function of $\mathcal{S}_k$. Then using (4.1.11) and (4.1.12) it can be shown that $\hat{\rho}_\varepsilon$ satisfies $\hat{\rho}_\varepsilon(a) = a$, $\hat{\rho}_\varepsilon(b) = b$. Also

$$\hat{\rho}'_\varepsilon(R) = \hat{\rho}'(R) \qquad \text{if } \hat{\rho}'(R) \notin \left(\frac{1}{k}, k\right). \tag{4.1.13}$$

Now $\hat{\rho}(a) = a > 0$, and so $\hat{\rho}_\varepsilon(R) > 0$ for $R \in (a,b)$ provided ε is sufficiently small, and from (4.1.13) we can say that $\hat{\rho}'_\varepsilon(R) > 0$ for a.e. $R \in (a,b)$ provided $\varepsilon < \dfrac{1}{2k \parallel v \parallel_\infty}$, and so $(\hat{\rho}_\varepsilon,\hat{\psi}) \in \mathcal{A}_N^{\text{sym}}$ for ε sufficiently small. Now, as

$$\tilde{g}\left(\left\{(\hat{\rho}')^2 + \left(\frac{\hat{\rho}}{R}\right)^2 + (\hat{\rho}\hat{\psi}')^2\right\}^{\frac{1}{2}}, \hat{\rho}'\frac{\hat{\rho}}{R}\right) = \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right), \tag{4.1.14}$$

we obtain, using the triangle inequality,

$$\begin{aligned}\left|\frac{1}{\varepsilon}\left\{\Psi\left(\hat{\rho}'_\varepsilon, \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\right\}\right| \leq{}& \left|\frac{1}{\varepsilon}\left\{\Psi\left(\hat{\rho}'_\varepsilon, \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right)\right\}\right| \\ &+ \left|\frac{1}{\varepsilon}\left\{\Psi\left(\hat{\rho}', \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\right\}\right|\end{aligned}$$

(4.1.15)

for $R \in (a,b)$. Now if $\hat{\rho}'(R) \in \left[\frac{1}{k}, k\right]$ then the right-hand side of (4.1.15) is bounded by a constant independent of ε. If $\hat{\rho}'(R) \notin \left[\frac{1}{k}, k\right]$ then the first term on the right-hand side of (4.1.15) is zero. We apply the mean value theorem to the second term on the right-hand side to get

$$\left|\frac{1}{\varepsilon}\left\{\Psi\left(\hat{\rho}', \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\right\}\right| = \\ \left|\Psi_{,2}(\hat{\rho}', \beta(R, \nu(R), \varepsilon), \hat{\psi}')\right| \frac{1}{R}\int_a^R v(\tau)\chi_k(\tau)d\tau,$$

where

$$\beta(R, \nu(R), \varepsilon) = \frac{\hat{\rho}(R)}{R} + \frac{\varepsilon\nu(R)}{R}\int_a^R v(\tau)\chi_k(\tau)d\tau$$

and $\nu(R) \in (0,1)$. Note that $\beta(R, \nu(R), \varepsilon)$ can also be written as

$$\beta(R, \nu(R), \varepsilon) = \frac{\hat{\rho}(R)}{R}\left\{\frac{1}{\hat{\rho}(R)}\left(\hat{\rho}(R) + \varepsilon\nu(R)\int_a^R v(\tau)\chi_k(\tau)d\tau\right)\right\}.$$

Now for ε sufficiently small we can say

$$\left|\frac{1}{\hat{\rho}}\left(\hat{\rho} + \varepsilon\nu(R)\int_a^R v(\tau)\chi_k(\tau)d\tau\right) - 1\right| < \delta.$$

Hence by (4.1.8) we can say that

$$\Psi_{,2}\left(\hat{\rho}', \frac{1}{\hat{\rho}}\left(\hat{\rho} + \varepsilon\nu(R)\int_a^R v(\tau)\chi_k(\tau)d\tau\right)\frac{\hat{\rho}}{R}, \hat{\psi}'\right)\frac{\hat{\rho}}{R} < M\left\{\Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}\right) + 1\right\}.$$

Hence

$$\left|\frac{1}{\varepsilon}\left\{\Psi\left(\hat{\rho}', \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\right\}\right| < \\ M\frac{R}{\hat{\rho}}\left\{\Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}\right) + 1\right\}\frac{1}{R}\int_a^R v(\tau)\chi_k(\tau)d\tau, \tag{4.1.16}$$

and, since $\hat{\rho}(R) > a$ for $R \in (a,b)$ and $I(\hat{\rho}, \hat{\psi}) < \infty$, we see that the right-hand side of (4.1.16) lies in $L^1(a,b)$.

Hence by the dominated convergence theorem,

$$0 = \lim_{\varepsilon \to 0} \int_a^b \frac{R}{\varepsilon} \left(\Psi\left(\hat{\rho}_\varepsilon', \frac{\hat{\rho}_\varepsilon}{R}, \hat{\psi}'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\right) dR =$$
$$\int_a^b R\left(\Psi_{,1}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\chi_k(R)v(R) + \frac{1}{R}\Psi_{,2}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\left(\int_a^R v(\tau)\chi_k(\tau)d\tau\right)\right) dR, \tag{4.1.17}$$

since $(\hat{\rho}, \hat{\psi})$ is the minimiser of I. Next observe that by (4.1.8), it follows that $\Psi_{,2}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}\right) \in L^1(a,b)$. Integration of (4.1.17) by parts gives us that

$$\int_{\mathcal{S}_k} \left\{ R\Psi_{,1}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right) + \int_R^b \Psi_{,2}\left(\hat{\rho}', \frac{\hat{\rho}}{\tau}, \hat{\psi}'\right) d\tau \right\} v(R) dR = 0. \tag{4.1.18}$$

Now as (4.1.18) holds for all $v \in L^\infty(a,b)$ with v satisfying (4.1.11), it follows that

$$R\Psi_{,1}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right) + \int_R^b \Psi_{,2}\left(\hat{\rho}', \frac{\hat{\rho}}{\tau}, \hat{\psi}'\right) d\tau = c_k \tag{4.1.19}$$

for a.e. $R \in \mathcal{S}_k$, where c_k is constant, and as the sets $\mathcal{S}_k$, $k \in (1, \infty)$, are nested, all the c_k's are equal (to c, say). Thus (4.1.19) holds a.e. in $[a,b]$.
Step 2. Now, for $\hat{\psi}$, let $w \in L^\infty(a,b)$ satisfy

$$\int_{\mathcal{S}_l} w(R) dR = 0 \tag{4.1.20}$$

and put

$$\hat{\psi}_\varepsilon(R) = \hat{\psi}(R) + \varepsilon \int_a^R w(\tau)\chi_l(\tau) d\tau, \tag{4.1.21}$$

where χ_l is the characteristic function of $\mathcal{S}_l$. From (4.1.20) and (4.1.21), $\hat{\psi}_\varepsilon(a) = 0$ and $\hat{\psi}_\varepsilon(b) = 2N\pi$ (with $N \in \mathbb{N}$). Hence $(\hat{\rho}, \hat{\psi}_\varepsilon) \in \mathcal{A}_N^{\text{sym}}$. Now, consider the term

$$\left| \frac{1}{\varepsilon} \left\{ \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}_\varepsilon'\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right) \right\} \right|. \tag{4.1.22}$$

If $\left|\hat{\psi}'(R)\right| < l$, then (4.1.22) is bounded by a constant independent of ε. If $\left|\hat{\psi}'(R)\right| \not< l$ then (4.1.22) is zero. Again,

$$\begin{aligned} 0 = \lim_{\varepsilon \to 0} \int_a^b \frac{R}{\varepsilon} \left(\Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'_\varepsilon\right) - \Psi\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right)\right) dR \\ = \int_{\mathcal{S}_l} R\Psi_{,3}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right) w(R) dR \end{aligned} \tag{4.1.23}$$

as $(\hat{\rho}, \hat{\psi})$ is a minimiser of I. As (4.1.23) holds for all $w \in L^\infty(a,b)$ satisfying (4.1.20), and as the sets $\mathcal{S}_l$, $l \in (0,\infty)$, are nested, it follows that

$$R\Psi_{,3}\left(\hat{\rho}', \frac{\hat{\rho}}{R}, \hat{\psi}'\right) = d \tag{4.1.24}$$

holds a.e. in $[a,b]$.

Step 3. We now want to show that $(\hat{\rho}, \hat{\psi}) \in C^1((a,b))$. We proceed in three parts:
Part (a). Putting $\Phi(R, \boldsymbol{v}, \boldsymbol{v}') = \Phi(R, (\rho, \psi), (\rho', \psi'))$ where

$$\Phi(R, \boldsymbol{v}, \boldsymbol{v}') := R\tilde{g}\left(\left|(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right|^{\frac{1}{2}}, \frac{\rho\rho'}{R}\right) = R\Psi\left(\rho', \frac{\rho}{R}, \psi'\right)$$

(and $\boldsymbol{v} = (\rho, \psi)$, $\boldsymbol{v}' = (\rho', \psi') \in \mathbb{R}^2$) and putting $\hat{\boldsymbol{v}} = (\hat{\rho}, \hat{\psi})$ and $\hat{\boldsymbol{v}}' = (\hat{\rho}', \hat{\psi}')$ we have

$$\begin{aligned}
\frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}') &:= \begin{pmatrix} \dfrac{\partial \Phi}{\partial \rho'}(R, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}')) \\ \\ \dfrac{\partial \Phi}{\partial \psi'}(R, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}')) \end{pmatrix} \\
&= \begin{pmatrix} R\Psi_{,1}\left(\hat{\rho}', \dfrac{\hat{\rho}}{R}, \hat{\psi}'\right) \\ R\Psi_{,3}\left(\hat{\rho}', \dfrac{\hat{\rho}}{R}, \hat{\psi}'\right) \end{pmatrix} = \begin{pmatrix} c - \displaystyle\int_R^b \Psi_{,2}\left(\hat{\rho}', \frac{\hat{\rho}}{t}, \hat{\psi}'\right) dt \\ d \end{pmatrix} \\
&= \begin{pmatrix} c - \displaystyle\int_R^b \frac{\partial \Phi}{\partial \rho}(t, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}'))dt \\ \\ d - \displaystyle\int_R^b \frac{\partial \Phi}{\partial \psi}(t, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}'))dt \end{pmatrix}
\end{aligned}$$

and so we have

$$\frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)) = -\int_R^b \frac{\partial \Phi}{\partial \boldsymbol{v}}(t, \hat{\boldsymbol{v}}(t), \hat{\boldsymbol{v}}'(t))dt + \boldsymbol{C} \tag{4.1.25}$$

for a.e. $R \in [a, b]$ where $\boldsymbol{C}$ is a constant vector of integration and

$$\frac{\partial \Phi}{\partial \boldsymbol{v}}(R, \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}') := \begin{pmatrix} \dfrac{\partial \Phi}{\partial \rho}(R, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}')) \\ \\ \dfrac{\partial \Phi}{\partial \psi}(R, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}')) \end{pmatrix}.$$

Thus, by (4.1.25), $\dfrac{\partial \Phi}{\partial \boldsymbol{v}'}$ is absolutely continuous on $[a, b]$.
Now as (4.1.25) holds for a.e. $R \in [a, b]$, we let $J \subset [a, b]$ be the subset of $[a, b]$ where (4.1.25) holds.
Now let $R \in [a, b] \setminus J$ and choose a sequence $(R_n) \subset J$ such that $R_n \to R$ as $n \to \infty$. Then we can write

$$-\int_{R_n}^b \frac{\partial \Phi}{\partial \boldsymbol{v}}(t, \hat{\boldsymbol{v}}(t), \hat{\boldsymbol{v}}'(t))dt + \boldsymbol{C} \to -\int_R^b \frac{\partial \Phi}{\partial \boldsymbol{v}}(t, \hat{\boldsymbol{v}}(t), \hat{\boldsymbol{v}}'(t))dt + \boldsymbol{C} = \boldsymbol{P}, \text{ say.} \tag{4.1.26}$$

Also, $\hat{\boldsymbol{v}}(R_n) \to \hat{\boldsymbol{v}}(R)$. We next show that $(\boldsymbol{v}'(R_n))$ is bounded.
Part (b). We do this by showing that $\left|\dfrac{\partial \Phi}{\partial \boldsymbol{v}'}\right| \to \infty$ as $|\boldsymbol{v}'| \to \infty$. Recalling that

$$\Phi(R, \boldsymbol{v}, \boldsymbol{v}') = R\tilde{g}\left(\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}, \rho'\frac{\rho}{R}\right),$$

and putting $S = \left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}$ and $T = \rho'\dfrac{\rho}{R}$, we can write

$$\frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \boldsymbol{v}, \boldsymbol{v}') = \begin{pmatrix} \dfrac{R\rho'}{S}\tilde{g}_{,1}(S,T) + \rho\tilde{g}_{,2}(S,T) \\ \\ \dfrac{R\rho^2\psi'}{S}\tilde{g}_{,1}(S,T) \end{pmatrix}. \tag{4.1.27}$$

Thus

$$\left|\frac{\partial \Phi}{\partial \boldsymbol{v}'}\right| = \left|\left(\frac{R\rho'}{S}\tilde{g}_{,1}(S,T) + \rho\tilde{g}_{,2}(S,T)\right)^2 + \left(\frac{R\rho^2\psi'}{S}\tilde{g}_{,1}(S,T)\right)^2\right|^{\frac{1}{2}}. \tag{4.1.28}$$

Now, we have $\tilde{g}_{,1}(S,T) > 0$, and $\tilde{g}(S,T) \to \infty$ as $T \to \infty$.
Also, by (R1) and (R2), $\tilde{g}_{,1}(S,T) \to \infty$ if and only if either $S \to \infty$ or $T \to \infty$, and $\tilde{g}_{,2}(S,T) \to -\infty$ if and only if $T \to 0$. Also, by convexity of $\tilde{g}$, we have $\tilde{g}_{,11}(S,T) \geq 0$ and $\tilde{g}_{,22}(S,T) \geq 0$.
Thus we can say that as $|(\rho', \psi')| \to \infty$,

$$\left|\left(\frac{R\rho'}{S}\tilde{g}_{,1}(S,T) + \rho\tilde{g}_{,2}(S,T)\right)^2 + \left(\frac{R\rho^2\psi'}{S}\tilde{g}_{,1}(S,T)\right)^2\right|^{\frac{1}{2}} \to \infty$$

and so $\left|\dfrac{\partial\Phi}{\partial \boldsymbol{v}'}\right| \to \infty$ as $|\boldsymbol{v}'| \to \infty$ uniformly. Thus $\left(\hat{\boldsymbol{v}}'(R_n)\right)_{n=1}^{\infty}$ is a bounded sequence, and hence contains a convergent subsequence, still labelled $\hat{\boldsymbol{v}}'(R_n)$, converging to $\hat{\boldsymbol{w}}$, say.
Therefore, we have that

$$\frac{\partial\Phi}{\partial \boldsymbol{v}'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R_n)) \to \frac{\partial\Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{w}}). \tag{4.1.29}$$

Hence, in $[a,b] \setminus J$ we have

$$\frac{\partial\Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{w}}) = -\int_R^b \frac{\partial\Phi}{\partial \boldsymbol{v}}(t, \hat{\boldsymbol{v}}(t), \hat{\boldsymbol{v}}'(t))dt + \boldsymbol{C} = \boldsymbol{P}, \text{ say.} \tag{4.1.30}$$

Also, the limit $\hat{\boldsymbol{w}}$ is unique, that is there is a unique value of $\hat{\boldsymbol{w}}$ such that (4.1.30) holds, since Φ is in $C^2(\mathbb{R}^5)$ and is such that

$$\det\left(\left(\frac{\partial^2\Phi}{\partial(\boldsymbol{v}')^2}\right)\right) = \frac{\partial^2\Phi}{\partial(\rho')^2}\frac{\partial^2\Phi}{\partial(\psi')^2} - \left(\frac{\partial^2\Phi}{\partial\rho'\partial\psi'}\right)^2 > 0. \tag{4.1.31}$$

Part (c). Now, the set $[a,b] \setminus J$ has zero measure and in this set we define $\hat{\boldsymbol{v}}'(R)$ such that (4.1.25) holds. Thus, for $R \in [a,b] \setminus J$, we define $\hat{\boldsymbol{w}} = \hat{\boldsymbol{v}}'(R)$ and show that $\hat{\boldsymbol{v}}'(R_n) \to \hat{\boldsymbol{v}}'(R)$.
Suppose (for a contradiction) that there exists a subsequence, still labelled (R_n), and $\varepsilon_0 > 0$ such that $\| \hat{\boldsymbol{v}}'(R_n) - \hat{\boldsymbol{v}}'(R) \| \geq \varepsilon_0$ for all n. Thus, we now want to show that

$$\left\|\frac{\partial\Phi}{\partial \boldsymbol{v}'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R_n)) - \frac{\partial\Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R))\right\| \geq \zeta > 0 \tag{4.1.32}$$

in order to force a contradiction.

Suppose (for a further contradiction) that there exists a further subsequence, still labelled (R_n), such that

$$\left\| \frac{\partial \Phi}{\partial \boldsymbol{v}'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R_n)) - \frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)) \right\| \to 0$$

as $n \to \infty$. Now, in index notation

$$\begin{aligned}
&\frac{\partial \Phi}{\partial v_i'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R_n)) - \frac{\partial \Phi}{\partial v_i'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)) \\
&\quad = \frac{\partial \Phi}{\partial v_i'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R_n)) - \frac{\partial \Phi}{\partial v_i'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R)) \\
&\qquad\qquad + \frac{\partial \Phi}{\partial v_i'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R)) - \frac{\partial \Phi}{\partial v_i'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)) \\
&\quad = \frac{\partial^2 \Phi}{\partial v_i' \partial v_j'}(R_n, \hat{\boldsymbol{v}}(R_n), \theta_n^i \hat{\boldsymbol{v}}'(R_n) + (1-\theta_n^i)\hat{\boldsymbol{v}}'(R))(\hat{v}_j'(R_n) - \hat{v}_j'(R)) \\
&\qquad\qquad + \frac{\partial \Phi}{\partial v_i'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R)) - \frac{\partial \Phi}{\partial v_i'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)),
\end{aligned}$$

where θ_n^i is such that $0 < \theta_n^i < 1$.
As $n \to \infty$, $\dfrac{\partial \Phi}{\partial \boldsymbol{v}'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R)) - \dfrac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)) \to 0$. Hence

$$\frac{\partial^2 \Phi}{\partial v_i' \partial v_j'}(R_n, \hat{\boldsymbol{v}}(R_n), \theta_n^i \hat{\boldsymbol{v}}'(R_n) + (1-\theta_n^i)\hat{\boldsymbol{v}}'(R))(\hat{v}_j'(R_n) - \hat{v}_j'(R)) \to 0$$

as $n \to \infty$, and thus

$$(\hat{v}_i'(R_n) - \hat{v}_i'(R))\frac{\partial^2 \Phi}{\partial v_i' \partial v_j'}(R_n, \hat{\boldsymbol{v}}(R_n), \theta_n^i \hat{\boldsymbol{v}}'(R_n) + (1-\theta_n^i)\hat{\boldsymbol{v}}'(R))(\hat{v}_j'(R_n) - \hat{v}_j'(R))$$

converges to 0 as $n \to \infty$ - a contradiction as the matrix $\left(\dfrac{\partial^2 \Phi}{\partial (\boldsymbol{v}')^2}\right)$ is positive definite.
Thus (4.1.32) holds - a contradiction as

$$\frac{\partial \Phi}{\partial \boldsymbol{v}'}(R_n, \hat{\boldsymbol{v}}(R_n), \hat{\boldsymbol{v}}'(R_n)) \to \frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}(R), \hat{\boldsymbol{v}}'(R)).$$

Thus, $\hat{\boldsymbol{v}}'(R_n) \to \hat{\boldsymbol{v}}'(R)$ as $n \to \infty$.
An argument similar to [8, Proposition 6.1] can be used to show that $\hat{\rho}' > 0$.
Step 4. We now show that $(\hat{\rho}, \hat{\psi}) \in C^2((a,b))$ The technique in showing this is based on the proof of [33, §1 Section 3.1 Proposition 3] pp 41-42. Putting

$\Phi(R, \boldsymbol{v}, \boldsymbol{v}') = \Phi(R, (\rho, \psi), (\rho', \psi'))$ where $\boldsymbol{v} = (\rho, \psi)$, $\boldsymbol{v}' = (\rho', \psi') \in \mathbb{R}^2$ and putting $\hat{\boldsymbol{v}} = (\hat{\rho}, \hat{\psi})$ and $\hat{\boldsymbol{v}}' = (\hat{\rho}', \hat{\psi}')$ as before we have that

$$\begin{aligned}
&\frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}') \\
&:= \begin{pmatrix} \dfrac{\partial \Phi}{\partial \rho'}(R, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}')) \\ \\ \dfrac{\partial \Phi}{\partial \psi'}(R, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}')) \end{pmatrix} = \begin{pmatrix} c - \displaystyle\int_R^b \frac{\partial \Phi}{\partial \rho}(t, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}'))dt \\ d \end{pmatrix}.
\end{aligned}$$

Thus, as $c - \displaystyle\int_R^b \frac{\partial \Phi}{\partial \rho}(t, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}'))dt$ and d are $C^1((a,b))$ we can put

$$\boldsymbol{\Pi}(R) := \begin{pmatrix} c - \displaystyle\int_R^b \frac{\partial \Phi}{\partial \rho}(t, (\hat{\rho}, \hat{\psi}), (\hat{\rho}', \hat{\psi}'))dt \\ d \end{pmatrix}$$

and thus we have

$$\frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}') - \boldsymbol{\Pi}(R) = \boldsymbol{0}.$$

Now we define a mapping $\boldsymbol{G} : (a,b) \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\boldsymbol{G}(R, \boldsymbol{v}') := \frac{\partial \Phi}{\partial \boldsymbol{v}'}(R, \hat{\boldsymbol{v}}, \boldsymbol{v}') - \boldsymbol{\Pi}(R).$$

Then $\boldsymbol{G} \in C^1((a,b) \times (0,\infty) \times \mathbb{R})$. Also, we define

$$\frac{\partial \boldsymbol{G}}{\partial \boldsymbol{v}'}(R, \boldsymbol{v}') := \begin{pmatrix} \Phi_{,\rho'\rho'}(R, (\hat{\rho}, \hat{\psi}), (\rho', \psi')) & \Phi_{,\rho'\psi'}(R, (\hat{\rho}, \hat{\psi}), (\rho', \psi')) \\ \Phi_{,\rho'\psi'}(R, (\hat{\rho}, \hat{\psi}), (\rho', \psi')) & \Phi_{,\psi'\psi'}(R, (\hat{\rho}, \hat{\psi}), (\rho', \psi')) \end{pmatrix}.$$

Hence, $\boldsymbol{G}$ satisfies $\det \left(\dfrac{\partial \boldsymbol{G}}{\partial \boldsymbol{v}'}(R, \boldsymbol{v}') \right) \neq 0$ for all $R \in (a,b)$. Also from the definition of $\boldsymbol{G}$ we can say that $\boldsymbol{G}(R, \hat{\boldsymbol{v}}') = \boldsymbol{0}$ for all $R \in (a,b)$. Hence, by the implicit function theorem, we can say that $(\hat{\rho}', \hat{\psi}') \in C^1((a,b))$, and hence $(\hat{\rho}, \hat{\psi}) \in C^2((a,b))$. Also, $(\hat{\rho}, \hat{\psi})$ satisfy the Euler-Lagrange equations

$$\frac{d}{dR}\left[R\Psi_{,1}\left(\rho', \frac{\rho}{R}, \psi'\right)\right] - \Psi_{,2}\left(\rho', \frac{\rho}{R}, \psi'\right) = 0 \tag{4.1.33}$$

and

$$\frac{d}{dR}\left[R\Psi_{,3}\left(\rho'\frac{\rho}{R},\psi'\right)\right]=0, \tag{4.1.34}$$

and, using (4.1.14), (4.1.33) and (4.1.34) can be rewritten in the form (4.1.9) and (4.1.10) respectively. $\square$

Remark 4.1.4 *One example of a function $\tilde{g}(S,T)$ satisfying (4.1.8) is $\tilde{g}(S,T)=\frac{1}{2}S^2+h(T)$, as considered in [57].*
Examples of functions $\tilde{g}(S,T)$ satisfying (R1) *and* (R2) *are those of the form $\tilde{g}(S,T)=f(S)+h(T)$ or $\tilde{g}(S,T)=(S+T)^2+h(T)$ (with f and h satisfying the conditions of Proposition 4.1.3).*

Finally, we state the relationship between solutions (ρ,ψ) of (4.1.9) and (4.1.10) and the solutions of the Euler-Lagrange equations for E.

Proposition 4.1.5 *Any smooth solution (ρ,ψ) of (4.1.9) and (4.1.10) gives rise to a corresponding solution $\boldsymbol{u}$ (given by (4.0.5)) of the full Euler-Lagrange equations for E (given by (4.0.3)), that is*

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla\boldsymbol{u}(\boldsymbol{x}))\right]=0,\ \boldsymbol{x}\in A,\ i=1,2, \tag{4.1.35}$$

for stored energy functions of the form (4.1.1).

Proof: We will first show that the weak form of the equilibrium equations, that is

$$\int_A \frac{\partial W}{\partial F}(\nabla\boldsymbol{u}(\boldsymbol{x})).\nabla\boldsymbol{\phi}(\boldsymbol{x})\ \mathrm{dx}=0, \tag{4.1.36}$$

hold for all $\boldsymbol{\phi}\in C^1_0(A)$. As $W(F)=\tilde{g}(|F|,\det(F))$ we have

$$\frac{\partial W}{\partial F}(F)=\tilde{g}_{,1}(|F|,\det(F))\frac{F}{|F|}+\tilde{g}_{,2}(|F|,\det(F))(\mathrm{adj}(F))^T.$$

Now put $F = \nabla \boldsymbol{u}$ where $\nabla \boldsymbol{u}$ is as given by (4.0.7). Then

$$
\begin{aligned}
F_1^1 = u^1_{,1} &= \{\rho' \cos(\theta+\psi) - \rho\psi' \sin(\theta+\psi)\}\cos(\theta) + \frac{\rho}{R}\sin(\theta+\psi)\sin(\theta),\\
F_2^1 = u^1_{,2} &= \{\rho' \cos(\theta+\psi) - \rho\psi' \sin(\theta+\psi)\}\sin(\theta) - \frac{\rho}{R}\sin(\theta+\psi)\cos(\theta),\\
F_1^2 = u^2_{,1} &= \{\rho' \sin(\theta+\psi) + \rho\psi' \cos(\theta+\psi)\}\cos(\theta) - \frac{\rho}{R}\cos(\theta+\psi)\sin(\theta),\\
F_2^2 = u^2_{,2} &= \{\rho' \sin(\theta+\psi) + \rho\psi' \cos(\theta+\psi)\}\sin(\theta) + \frac{\rho}{R}\cos(\theta+\psi)\cos(\theta).
\end{aligned}
\tag{4.1.37}
$$

Also putting $W(\nabla \boldsymbol{u}) = \bar{W}\left(\rho', \frac{\rho}{R}, \rho\psi'\right)$ gives

$$
\begin{aligned}
\frac{\partial \bar{W}}{\partial \rho'} &= \frac{\partial W}{\partial F_1^1}(\nabla \boldsymbol{u})\frac{\partial u^1_{,1}}{\partial \rho'} + \frac{\partial W}{\partial F_2^1}(\nabla \boldsymbol{u})\frac{\partial u^1_{,2}}{\partial \rho'} + \frac{\partial W}{\partial F_1^2}(\nabla \boldsymbol{u})\frac{\partial u^2_{,1}}{\partial \rho'} + \frac{\partial W}{\partial F_2^2}(\nabla \boldsymbol{u})\frac{\partial u^2_{,2}}{\partial \rho'},\\
\frac{\partial \bar{W}}{\partial \left(\frac{\rho}{R}\right)} &= \frac{\partial W}{\partial F_1^1}(\nabla \boldsymbol{u})\frac{\partial u^1_{,1}}{\partial \left(\frac{\rho}{R}\right)} + \frac{\partial W}{\partial F_2^1}(\nabla \boldsymbol{u})\frac{\partial u^1_{,2}}{\partial \left(\frac{\rho}{R}\right)}\\
&\qquad + \frac{\partial W}{\partial F_1^2}(\nabla \boldsymbol{u})\frac{\partial u^2_{,1}}{\partial \left(\frac{\rho}{R}\right)} + \frac{\partial W}{\partial F_2^2}(\nabla \boldsymbol{u})\frac{\partial u^2_{,2}}{\partial \left(\frac{\rho}{R}\right)},\\
\frac{\partial \bar{W}}{\partial (\rho\psi')} &= \frac{\partial W}{\partial F_1^1}(\nabla \boldsymbol{u})\frac{\partial u^1_{,1}}{\partial (\rho\psi')} + \frac{\partial W}{\partial F_2^1}(\nabla \boldsymbol{u})\frac{\partial u^1_{,2}}{\partial (\rho\psi')}\\
&\qquad + \frac{\partial W}{\partial F_1^2}(\nabla \boldsymbol{u})\frac{\partial u^2_{,1}}{\partial (\rho\psi')} + \frac{\partial W}{\partial F_2^2}(\nabla \boldsymbol{u})\frac{\partial u^2_{,2}}{\partial (\rho\psi')},
\end{aligned}
\tag{4.1.38}
$$

and so from (4.1.37) and (4.1.38) we have

$$
\begin{aligned}
\frac{\partial \bar{W}}{\partial \rho'} &= \frac{\partial W}{\partial F_1^1}\cos(\theta+\psi)\cos(\theta) + \frac{\partial W}{\partial F_2^1}\cos(\theta+\psi)\sin(\theta)\\
&\qquad + \frac{\partial W}{\partial F_1^2}\sin(\theta+\psi)\cos(\theta) + \frac{\partial W}{\partial F_2^2}\sin(\theta+\psi)\sin(\theta),\\
\frac{\partial \bar{W}}{\partial \left(\frac{\rho}{R}\right)} &= \frac{\partial W}{\partial F_1^1}\sin(\theta+\psi)\sin(\theta) - \frac{\partial W}{\partial F_2^1}\sin(\theta+\psi)\cos(\theta)\\
&\qquad - \frac{\partial W}{\partial F_1^2}\cos(\theta+\psi)\sin(\theta) + \frac{\partial W}{\partial F_2^2}\sin(\theta+\psi)\sin(\theta),\\
\frac{\partial \bar{W}}{\partial (\rho\psi')} &= -\frac{\partial W}{\partial F_1^1}\sin(\theta+\psi)\cos(\theta) - \frac{\partial W}{\partial F_2^1}\sin(\theta+\psi)\sin(\theta)\\
&\qquad + \frac{\partial W}{\partial F_1^2}\cos(\theta+\psi)\cos(\theta) + \frac{\partial W}{\partial F_2^2}\cos(\theta+\psi)\sin(\theta),
\end{aligned}
$$

and thus it can be shown that

$$V := \begin{pmatrix} \dfrac{\partial \bar{W}}{\partial (\rho')} & 0 \\ \\ \dfrac{\partial \bar{W}}{\partial (\rho\psi')} & \dfrac{\partial \bar{W}}{\partial (\frac{\rho}{R})} \end{pmatrix} =$$

$$\begin{pmatrix} \cos(\theta+\psi) & \sin(\theta+\psi) \\ -\sin(\theta+\psi) & \cos(\theta+\psi) \end{pmatrix} \begin{pmatrix} \dfrac{\partial W}{\partial F_1^1}(\nabla \boldsymbol{u}) & \dfrac{\partial W}{\partial F_2^1}(\nabla \boldsymbol{u}) \\ \\ \dfrac{\partial W}{\partial F_1^2}(\nabla \boldsymbol{u}) & \dfrac{\partial W}{\partial F_2^2}(\nabla \boldsymbol{u}) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$= P^T \frac{\partial W}{\partial F} Q, \text{ say.} \tag{4.1.39}$$

Also, putting

$$\boldsymbol{\phi}(\boldsymbol{x}) = \phi_1(R,\theta) \begin{pmatrix} \cos(\theta+\psi(R)) \\ \sin(\theta+\psi(R)) \end{pmatrix} + \phi_2(R,\theta) \begin{pmatrix} -\sin(\theta+\psi(R)) \\ \cos(\theta+\psi(R)) \end{pmatrix}, \tag{4.1.40}$$

where $\psi(R)$ is such that $I(\rho,\psi) = \inf_{\mathcal{A}_N^{\text{sym}}} I$ and $\phi_1(R,\theta)$, $\phi_2(R,\theta)$ are continuous in R and θ with continuous derivatives and satisfy $\phi_1(a,\theta) = \phi_1(b,\theta) = 0$ and $\phi_2(a,\theta) = \phi_2(b,\theta) = 0$ for all $\theta \in [0, 2\pi)$ (since $\boldsymbol{\phi}(\boldsymbol{x}) = \mathbf{0}$ on ∂A) and $\phi_1(R,0) = \phi_1(R,2\pi)$ and $\phi_2(R,0) = \phi_2(R,2\pi)$ for all $R \in [a,b]$, we obtain

$$\nabla \boldsymbol{\phi} = \begin{pmatrix} \cos(\theta+\psi) & -\sin(\theta+\psi) \\ \sin(\theta+\psi) & \cos(\theta+\psi) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \phi_1}{\partial R} & \dfrac{1}{R}\dfrac{\partial \phi_1}{\partial \theta} \\ \\ \phi_1\psi' & \dfrac{\phi_1}{R} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$+ \begin{pmatrix} -\sin(\theta+\psi) & -\cos(\theta+\psi) \\ \cos(\theta+\psi) & -\sin(\theta+\psi) \end{pmatrix} \begin{pmatrix} \dfrac{\partial \phi_2}{\partial R} & \dfrac{1}{R}\dfrac{\partial \phi_2}{\partial \theta} \\ \\ \phi_2\psi' & \dfrac{\phi_2}{R} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$= (PB + MC)Q^T, \text{ say.} \tag{4.1.41}$$

Now $\dfrac{\partial W}{\partial F}$ is of the form PVQ^T and $\nabla\phi$ is of the form $(PB+MC)Q^T$. Also

$$\begin{aligned}\frac{\partial W}{\partial F}\cdot\nabla\phi =& \mathrm{tr}\left(\frac{\partial W}{\partial F}(\nabla\phi)^T\right)\\ &= \mathrm{tr}(PVQ^TQ(B^TP^T+C^TM^T)) = \mathrm{tr}\left(VB^T+\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}VC^T\right)\end{aligned}$$

since $PP^T=\mathrm{Id}=Q^TQ$ and

$$M^TP=$$
$$\begin{pmatrix}-\sin(\theta+\psi) & \cos(\theta+\psi)\\ -\cos(\theta+\psi) & -\sin(\theta+\psi)\end{pmatrix}\begin{pmatrix}\cos(\theta+\psi) & -\sin(\theta+\psi)\\ \sin(\theta+\psi) & \cos(\theta+\psi)\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Also, for $W(F)=\tilde{g}(|F|,\det(F))$ we have that

$$\begin{aligned}&\frac{\partial\bar{W}}{\partial(\rho')}=\frac{\rho'}{S}\tilde{g}_{,1}(S,T)+\frac{\rho}{R}\tilde{g}_{,2}(S,T),\ \frac{\partial\bar{W}}{\partial(\rho\psi')}=\frac{\rho\psi'}{S}\tilde{g}_{,1}(S,T)=\frac{1}{\rho}\frac{\partial\bar{W}}{\partial\psi'}\\ &\frac{\partial\bar{W}}{\partial\rho}=\frac{1}{S}\tilde{g}_{,1}(S,T)\left\{\frac{\rho}{R^2}+\rho(\psi')^2\right\}+\frac{\rho'}{R}\tilde{g}_{,2}(S,T)=\frac{1}{R}\frac{\partial\bar{W}}{\partial\left(\frac{\rho}{R}\right)},\end{aligned}$$

where $S=\left\{(\rho')^2+\left(\frac{\rho}{R}\right)^2+(\rho\psi')^2\right\}^{\frac{1}{2}}$ and $T=\rho'\dfrac{\rho}{R}$. Now, by direct calculation we have that

$$\begin{aligned}\frac{\partial W}{\partial F}\cdot\nabla\phi=&\left\{\frac{\rho'}{S}\tilde{g}_{,1}+\frac{\rho}{R}\tilde{g}_{,2}\right\}\frac{\partial\phi_1}{\partial R}+\left\{\frac{1}{S}\left[\frac{\rho}{R^2}+\rho(\psi')^2\right]\tilde{g}_{,1}+\frac{\rho'}{R}\tilde{g}_{,2}\right\}\phi_1\\ &+\left\{\frac{\rho\psi'}{S}\tilde{g}_{,1}\right\}\frac{\partial\phi_2}{\partial R}-\frac{\rho'\psi'}{S}\tilde{g}_{,1}\phi_2-\rho\psi'\frac{\partial\phi_2}{\partial\theta}+\left\{\frac{\rho}{R^2S}+\rho'\right\}\tilde{g}_{,1}\frac{\partial\phi_1}{\partial\theta}.\end{aligned}$$

Hence the first variation is

$$\begin{aligned}&\int_A\frac{\partial W}{\partial F^i_\alpha}(\nabla\boldsymbol{u})\frac{\partial\phi_i}{\partial x_\alpha}(\boldsymbol{x})\ \mathrm{dx}=\\ &\int_{R=a}^{b}\int_{\theta=0}^{2\pi}R\left\{\frac{\rho'}{S}\tilde{g}_{,1}+\frac{\rho}{R}\tilde{g}_{,2}\right\}\frac{\partial\phi_1}{\partial R}+\left\{\frac{1}{S}\left[\frac{\rho}{R}+R\rho(\psi')^2\right]\tilde{g}_{,1}+\rho'\tilde{g}_{,2}\right\}\phi_1\\ &+\left\{\frac{R\rho\psi'}{S}\tilde{g}_{,1}\right\}\frac{\partial\phi_2}{\partial R}-\frac{R\rho'\psi'}{S}\tilde{g}_{,1}\phi_2-R\rho\psi'\frac{\partial\phi_2}{\partial\theta}+\left\{\frac{\rho}{RS}+R\rho'\right\}\tilde{g}_{,1}\frac{\partial\phi_1}{\partial\theta}\ dR\ d\theta,\end{aligned}\tag{4.1.42}$$

and, as $\rho(R)$ and $\psi(R)$ are independent of θ, by integration by parts with respect to θ (4.1.42) becomes

$$\int_A \frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u})\frac{\partial \phi_i}{\partial x_\alpha}(\boldsymbol{x})\,\mathrm{dx} =$$

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{\frac{\rho'}{S}\tilde{g}_{,1} + \frac{\rho}{R}\tilde{g}_{,2}\right\}\frac{\partial \phi_1}{\partial R} + \left\{\frac{1}{S}\left[\frac{\rho}{R} + R\rho(\psi')^2\right]\tilde{g}_{,1} + \rho'\tilde{g}_{,2}\right\}\phi_1$$

$$+\left\{\frac{R\rho\psi'}{S}\tilde{g}_{,1}\right\}\frac{\partial \phi_2}{\partial R} - \frac{R\rho'\psi'}{S}\tilde{g}_{,1}\phi_2\; dR\, d\theta. \tag{4.1.43}$$

As $(\rho, \psi) \in C^2((a,b))$ by integrating by parts with respect to R we can write (4.1.43) as

$$\int_A \frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u})\frac{\partial \phi_i}{\partial x_\alpha}(\boldsymbol{x})\,\mathrm{dx} =$$

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi}\left\{-\frac{d}{dR}\left[R\left\{\frac{\rho'}{S}\tilde{g}_{,1} + \frac{\rho}{R}\tilde{g}_{,2}\right\}\right] + \frac{1}{S}\left\{\frac{\rho}{R} + R\rho(\psi')^2\right\}\tilde{g}_{,1} + \rho'\tilde{g}_{,2}\right\}\phi_1$$

$$-\left\{\frac{d}{dR}\left[\frac{R\rho\psi'}{S}\tilde{g}_{,1}\right] + \frac{R\rho'\psi'}{S}\tilde{g}_{,1}\right\}\phi_2\; dR\, d\theta. \tag{4.1.44}$$

Now on noting that

$$\frac{d}{dR}\left[\frac{R\rho\psi'}{S}\tilde{g}_{,1}\right] + \frac{R\rho'\psi'}{S}\tilde{g}_{,1} = \frac{1}{\rho}\frac{d}{dR}\left[\frac{R\rho^2\psi'}{S}\tilde{g}_{,1}\right],$$

it follows that

$$\int_A \frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u})\frac{\partial \phi_i}{\partial x_\alpha}(\boldsymbol{x})\,\mathrm{dx} =$$

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi}\left\{-\frac{d}{dR}\left[R\left\{\frac{\rho'}{S}\tilde{g}_{,1} + \frac{\rho}{R}\tilde{g}_{,2}\right\}\right] + \frac{1}{S}\left\{\frac{\rho}{R} + R\rho(\psi')^2\right\}\tilde{g}_{,1} + \rho'\tilde{g}_{,2}\right\}\phi_1$$

$$-\left\{\frac{1}{\rho}\frac{d}{dR}\left[\frac{R\rho^2\psi'}{S}\tilde{g}_{,1}\right]\right\}\phi_2\; dR\, d\theta. \tag{4.1.45}$$

However, the coefficients of ϕ_1 and ϕ_2 in (4.1.45) are the expressions on the left-hand side of the Euler-Lagrange equations (4.1.9) and (4.1.10) respectively, and so a solution (ρ, ψ) satisfying (4.1.9) and (4.1.10) gives rise to a corresponding solution $\boldsymbol{u}$ of the form (4.0.5) which satisfies

$$\int_A \frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u}) \frac{\partial \phi_i}{\partial x_\alpha}(\boldsymbol{x}) \mathrm{dx} = 0. \tag{4.1.46}$$

As (4.1.46) holds for all $\boldsymbol{\phi}$, then by the fundamental lemma of the calculus of variations (Lemma 3.1.22), integration of (4.1.45) by parts results in $\boldsymbol{u}$ satisfying the equation (4.1.35) as required. $\square$

It can also be shown, by a similar argument, that if $\boldsymbol{u}$ (given by (4.0.5)) is a solution of the full equilibrium equations, then (ρ, ψ) satisfy (4.1.9) and (4.1.10). We also note the following.

Corollary 4.1.6 *If (ρ, ψ) is a minimiser of I on $\mathcal{A}_N^{\mathrm{sym}}$, then ψ is monotonic on $[a, b]$.*

Proof: By Proposition 4.1.3, a minimiser (ρ, ψ) of I on $\mathcal{A}_N^{\mathrm{sym}}$ must satisfy the second Euler-Lagrange equation

$$\frac{d}{dR}\left[\frac{R\rho^2\psi'}{S}\tilde{g}_{,1}\right] = 0,$$

where $S = |\nabla \boldsymbol{u}|$. Hence, we can write

$$\psi' = \frac{kS}{R\rho^2 \tilde{g}_{,1}}$$

and as $\tilde{g}_{,1} > 0$ we have that ψ is monotonic on $[a, b]$. $\square$

Graphical representation of solutions for certain stored energy functions.

The software package AUTO can be used to give a graphical representation of solutions $\rho(R)$ and $\psi(R)$ to the rotationally symmetric Euler-Lagrange equations for E of the form (4.0.9), the equations being

$$\begin{aligned} &\frac{d}{dR}\left[R\rho'(R) + \rho(R)h'\left(\rho'(R)\frac{\rho(R)}{R}\right)\right] \\ &\quad = \frac{\rho(R)}{R} + R\rho(R)(\psi'(R))^2 + \rho'(R)h'\left(\rho'(R)\frac{\rho(R)}{R}\right) \end{aligned} \tag{4.1.47}$$

and

$$\frac{d}{dR}\left[R\rho^2(R)\psi'(R)\right] = 0, \tag{4.1.48}$$

for specified functions $h(d)$. AUTO was developed, initially by Doedel, to perform limited numerical bifurcation analysis on algebraic systems and systems of first order ordinary differential equations of the form $u'(t) = f(u(t), p)$ where $f(\cdot, \cdot)$, $u(\cdot) \in \mathbb{R}^n$ and p denotes one or more free parameters.
With this in mind, we must rewrite the two Euler-Lagrange equations as a system of four first order differential equations. In order to do this we require an explicit form for $W(F)$. We will put $W(F) = \frac{1}{2}|F|^2 + h(\det(F))$, and the examples we will consider have h of the form

(i) $h(d) = d - \log(d)$,

(ii) $h(d) = d + \dfrac{1}{2d}$,

(iii) $h(d) = d + \dfrac{1}{6d^2}$.

Now in the case where $F = \nabla \boldsymbol{u}$, where $\boldsymbol{u}(\boldsymbol{x})$ is given by (4.0.5), and

$$W(\nabla \boldsymbol{u}) = \frac{1}{2}|\nabla \boldsymbol{u}|^2 + \det(\nabla \boldsymbol{u}) + \frac{1}{n(n+1)(\det(\nabla \boldsymbol{u}))^n}$$

(with $n > 0$) the two Euler-Lagrange equations are

$$\begin{aligned}\rho'(R) &+ R\rho''(R) - \frac{\rho(R)}{R} - R\rho(R)(\psi'(R))^2 \\ &+ \rho(R)\left(\frac{(\rho'(R))^2}{R} + \frac{\rho(R)\rho''(R)}{R} - \frac{\rho(R)\rho'(R)}{R^2}\right)\left(\frac{R}{\rho(R)\rho'(R)}\right)^{n+2} = 0\end{aligned} \tag{4.1.49}$$

and

$$\rho^2(R)\psi'(R) + 2R\rho(R)\rho'(R)\psi'(R) + R\rho^2(R)\psi''(R) = 0. \tag{4.1.50}$$

(A similar form occurs for when $h(d) = d - \log(d)$ and in that case $n = 0$.) Hence (4.1.49) and (4.1.50) can be rewritten as

$$\rho''(R) = \frac{\dfrac{\rho(R)}{R} + R\rho(R)(\psi'(R))^2 - \rho'(R) - \left(\dfrac{R}{\rho(R)\rho'(R)}\right)^{n+1}\left\{\rho'(R) - \dfrac{\rho(R)}{R}\right\}}{R + \dfrac{R}{(\rho'(R))^2}\left(\dfrac{R}{\rho(R)\rho'(R)}\right)^n} \tag{4.1.51}$$

and

$$\psi''(R) = -\frac{[\rho(R)\psi'(R) + 2R\rho'(R)\psi'(R)]}{R\rho(R)} \tag{4.1.52}$$

Now (4.1.51) can be rearranged as

$$\rho'' = \rho' \left\{ \frac{\dfrac{(\rho\rho')^{n+1}}{R^2} + (\rho\rho')^{n+1}(\psi')^2 - \dfrac{\rho^n(\rho')^{n+2}}{R} - \dfrac{R^n\rho'}{\rho} + R^{n-1}}{\rho^n(\rho')^{n+1} + R^n} \right\}. \tag{4.1.53}$$

Now (4.1.52) and (4.1.53) can be rewritten as a system of first order equations by putting

$$\begin{aligned}
\rho' &= \sigma \\
\psi' &= \tau \\
\sigma' &= \sigma \left\{ \frac{\dfrac{(\rho\sigma)^{n+1}}{R^2} + (\rho\sigma)^{n+1}\tau^2 - \dfrac{\rho^n\sigma^{n+2}}{R} - \dfrac{R^n\sigma}{\rho} + R^{n-1}}{\rho^n(\sigma)^{n+1} + R^n} \right\} \\
\tau' &= -\frac{(\rho\tau + 2R\sigma\tau)}{R\rho}
\end{aligned}$$

(where σ and τ are functions of R). Using AUTO plots can be obtained for particular solutions ρ and ψ of this system of equations for each of the stored energy function $W(F) = \frac{1}{2}|F|^2 + h(\det(F))$ for h given by (i)-(iii) for twists of 2π to 8π, and using Matlab, with the data from AUTO, the radial line of the annulus for twists of 2π and 4π of the boundary can be plotted. These are as shown. (In the plots we are putting $a = 2$ and $b = 3$.)

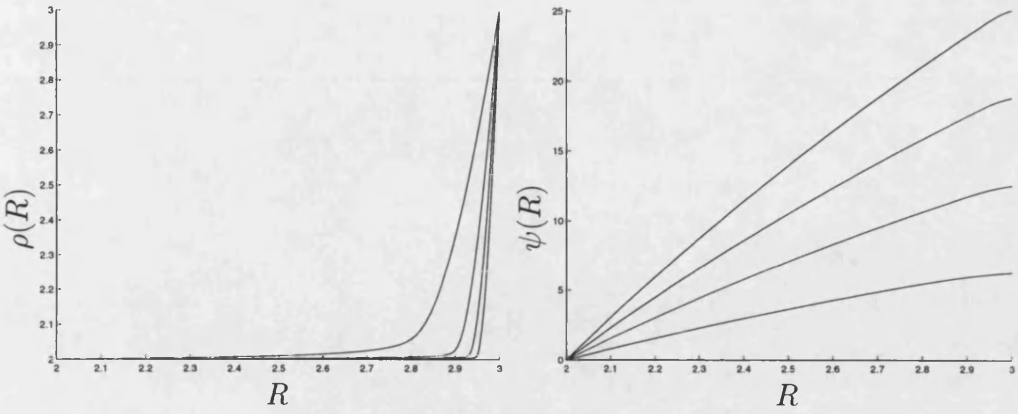

Figure 4.1: Graphs of ρ and ψ for $h(d) = d - \log(d)$ for one to four twists.

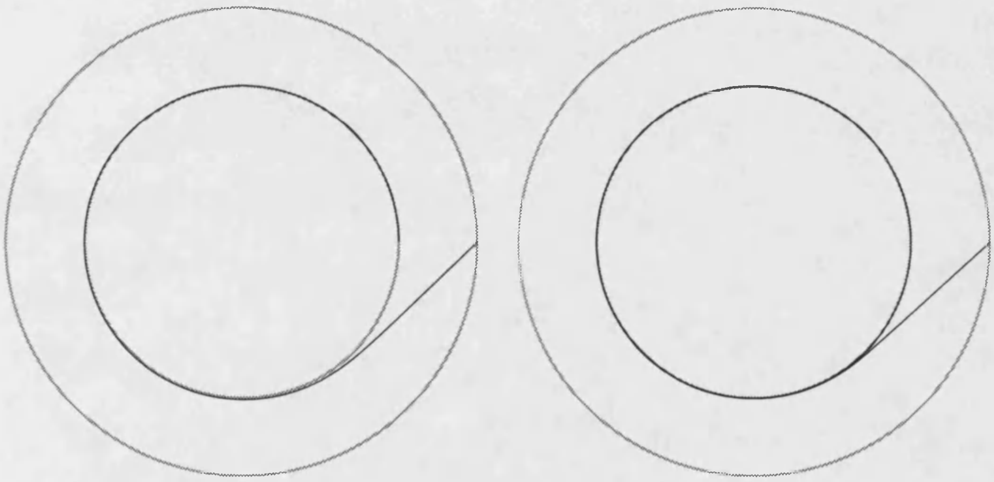

Figure 4.2: The radial line of the annulus for the case $h(d) = d - \log(d)$ for one and two twists.

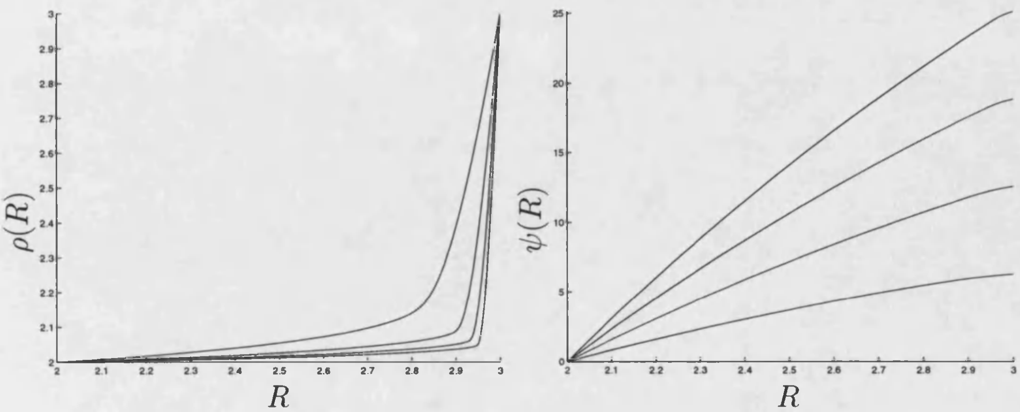

Figure 4.3: Graphs of ρ and ψ for $h(d) = d + \dfrac{1}{2d}$ for one to four twists.

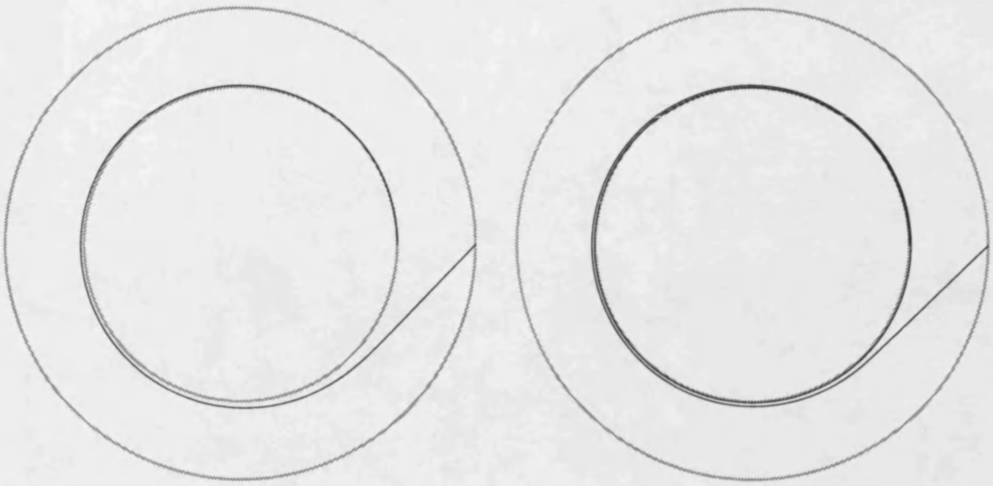

Figure 4.4: The radial line of the annulus for the case $h(d) = d + \dfrac{1}{2d}$ for one and two twists.

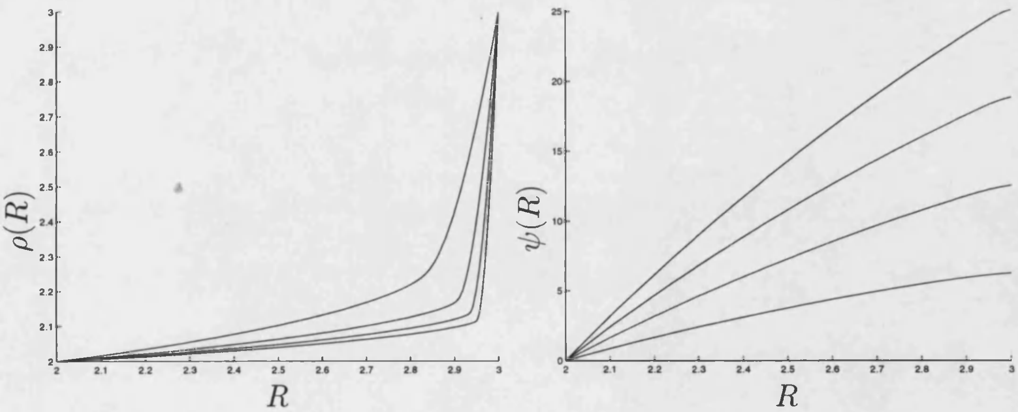

Figure 4.5: Graphs of ρ and ψ for $h(d) = d + \dfrac{1}{6d^2}$ for one to four twists.

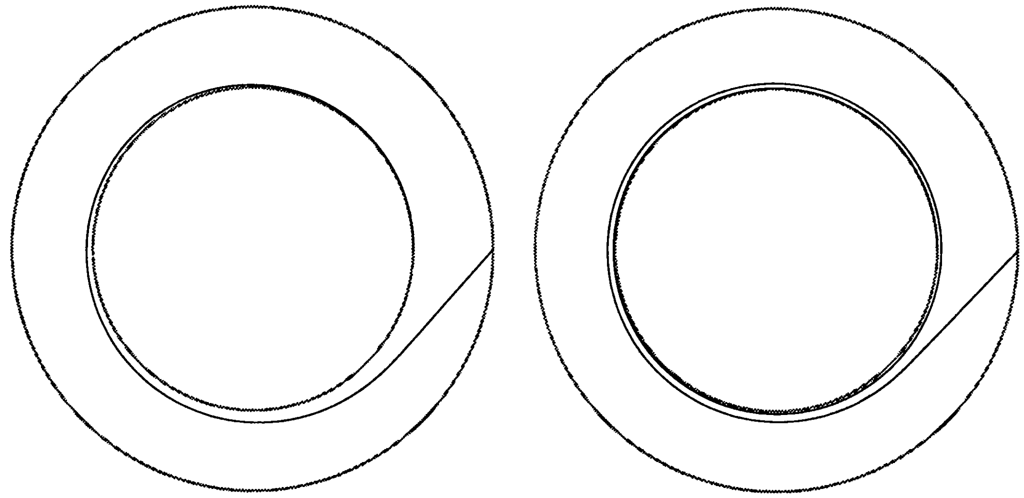

Figure 4.6: The radial line of the annulus for the case $h(d) = d + \dfrac{1}{6d^2}$ for one and two twists.

4.2 The degenerate case: explicit determination of a potential minimiser.

In this section we consider a model annulus problem in which the energy functional is such that the Euler-Lagrange equations can be integrated to obtain an explicit solution. For our model problem we consider W of the form (4.0.6) and put $h \equiv 0$. As a consequence the stored energy function is $W(F) = \dfrac{1}{2}|F|^2$, and so we are minimising the Dirichlet integral: we refer to this case as the *degenerate case* as the stored energy function no longer satisfies $W(F) \to \infty$ as $\det(F) \to 0_+$. Restricting ourselves to deformations $\boldsymbol{u}$ of the form (4.0.5) the energy functional is

$$E(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}|^2 \mathrm{dx} = \pi \int_{R=a}^{b} R\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\} dR =: 2\pi \bar{I}(\rho,\psi) \tag{4.2.1}$$

and the Euler-Lagrange equations for $\bar{I}(\rho,\psi)$ given by (4.2.1) are formally

$$\frac{d}{dR}\left[R\rho'(R)\right] = \frac{\rho(R)}{R} + R\rho(R)(\psi'(R))^2, \tag{4.2.2}$$

$$\frac{d}{dR}\left[R\rho^2(R)\psi'(R)\right] = 0. \tag{4.2.3}$$

We will define a suitable set of admissible deformations for the model problem of minimising $E(\boldsymbol{u})$ as given by (4.2.1) and we will obtain a candidate for a minimiser (ρ, ψ) on that set. It will be shown later (Chapter 6) that this candidate minimises the Dirichlet integral in a more general class of admissible variations.
We first determine an admissible set of deformations on which to minimise E.
As $W(F)$ is strictly convex there is only one solution of the Euler-Lagrange equations throughout A satisfying the boundary conditions $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ on ∂A and $\det(\nabla \boldsymbol{u}) > 0$ and this is the identity map $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ in A. Hence if $N > 0$ there will not be a minimiser (ρ, ψ) of $\bar{I}(\rho, \psi)$ on the set

$$\begin{aligned}\mathcal{A}_N^{\text{sym}} = \{(\rho, \psi) \in W^{1,1}((a,b)) : \rho(a) = a, \rho(b) = b, \rho'(R) > 0 \text{ a.e. on } (a,b),\\ \psi(a) = 0, \psi(b) = 2N\pi\}.\end{aligned}$$

However, the technique of [64] can be adapted to show the existence of minimisers (ρ, ψ) of $\bar{I}(\rho, \psi)$ on the set

$$\begin{aligned}\bar{\mathcal{A}}_N^{\text{sym}} = \{(\rho, \psi) \in W^{1,1}((a,b)) : \rho(a) = a, \rho(b) = b, \rho'(R) \geq 0 \text{ a.e. on } (a,b),\\ \psi(a) = 0, \psi(b) = 2N\pi\}.\end{aligned}$$

Hence we will consider $\bar{\mathcal{A}}_N^{\text{sym}}$ as our set of admissible deformations. Thus we are allowing deformations $\boldsymbol{u}$ such that on part of the annulus $\det(\nabla \boldsymbol{u})$ can equal 0. (Note that we are not allowed to take two-sided variations at those points of a minimiser where $\det(\nabla \boldsymbol{u}(\boldsymbol{x})) = 0$.)
Now a minimiser for $\bar{I}(\rho, \psi)$ can only satisfy both Euler-Lagrange equations in the region where $\det(\nabla \boldsymbol{u}) > 0$. Hence the function ρ corresponding to the minimiser (ρ, ψ) will not satisfy the Euler-Lagrange equation for ρ, that is (4.2.2), throughout $[a, b]$, since on part of the annulus we must have $\rho'(R) = 0$ as $\det(\nabla \boldsymbol{u}) = \rho' \dfrac{\rho}{R}$.
However, the condition $\det(\nabla \boldsymbol{u}) \geq 0$ does not prevent us from taking two-sided variations of ψ around a minimiser. Thus the Euler-Lagrange equation for ψ, that is (4.2.3), is satisfied throughout $[a, b]$.
Now we can rewrite (4.2.3) as

$$\psi'(R) = \frac{k}{R\rho^2(R)}, \tag{4.2.4}$$

where k is a constant of integration, and hence (4.2.2) becomes

$$R\rho''(R) + \rho'(R) - \frac{\rho(R)}{R} - \frac{k^2}{R\rho^3(R)} = 0. \tag{4.2.5}$$

A possible candidate for $\rho(R)$ can be constructed by smoothly piecing together a solution $\hat{\rho}$ of the equation (4.2.5) on $[R_0, b]$ with a constant function on $[a, R_0]$, where R_0 is chosen so that $\hat{\rho}'(R_0) = 0$ and $\hat{\rho}'(R) > 0$ for $R > R_0$. Hence, our candidate for $\rho(R)$ is

$$\rho(R) = \begin{cases} a & \text{on } [a, R_0] \\ \hat{\rho}(R) & \text{on } [R_0, b]. \end{cases} \tag{4.2.6}$$

Also, our candidate for $\psi(R)$ is

$$\psi(R) = \begin{cases} \hat{\psi}_1(R) & \text{on } [a, R_0] \\ \hat{\psi}_2(R) & \text{on } [R_0, b]. \end{cases} \tag{4.2.7}$$

where $\hat{\psi}_1(R)$ is such that (4.2.4) holds on $[a, R_0]$ and $\hat{\psi}_2(R)$ is such that (4.2.4) holds on $[R_0, b]$.
The main result in this section is the following.

Proposition 4.2.1 *Let $(\rho, \psi) \in \bar{\mathcal{A}}_N^{\mathrm{sym}}$ with $\rho(R)$ defined by (4.2.6) and $\psi(R)$ defined by (4.2.7) be the form for the minimiser of $\bar{I}(\rho, \psi)$ on $\bar{\mathcal{A}}_N^{\mathrm{sym}}$, with $\hat{\rho}(R)$ satisfying (4.2.5) and $\psi(R)$ satisfying (4.2.4), and where R_0 is such that $\rho'(R_0) = 0$, $\rho(R_0) = a$ and $\rho(R) > 0$ for $R > R_0$. Then our candidate for an explicit form of a solution to the problem of minimsing $\bar{I}(\rho, \psi)$ on $\bar{\mathcal{A}}_N^{\mathrm{sym}}$ is*

$$\rho(R) = \begin{cases} a & \textit{on } [a, R_0] \\ \dfrac{1}{\sqrt{2}} \left\{ \hat{A}R^2 + \dfrac{\hat{B}}{\hat{A}R^2} - \hat{C} \right\}^{\frac{1}{2}} & \textit{on } [R_0, b] \end{cases} \tag{4.2.8}$$

and

$$\psi(R) = \begin{cases} \dfrac{k}{a^2} \log\left(\dfrac{R}{a}\right) & \textit{on } [a, R_0] \\ \tan^{-1}\left(\dfrac{2\hat{A}R^2 - \hat{C}^2}{2k}\right) - \tan^{-1}\left(\dfrac{2\hat{A}b^2 - \hat{C}^2}{2k}\right) + 2N\pi & \textit{on } [R_0, b], \end{cases} \tag{4.2.9}$$

where

$$\hat{A} = \frac{1}{b^2}\left\{b^2 + \frac{1}{2}\left(\frac{k^2}{a^2}\right) + \left[\frac{1}{a^2}(b^2 - a^2)(a^2b^2 + k^2)\right]^{\frac{1}{2}}\right\}, \quad \hat{B} = \frac{1}{4}\left(a^2 + \frac{k^2}{a^2}\right)^2$$

and $\hat{C} = \dfrac{k^2}{a^2} - a^2$, *and where* k *is a constant of integration determined by solving*

$$\frac{k}{a^2}\log\left(\frac{R_0}{a}\right) = \tan^{-1}\left(\frac{2\hat{A}R_0^2 - \hat{C}^2}{2k}\right) - \tan^{-1}\left(\frac{2\hat{A}b^2 - \hat{C}^2}{2k}\right) + 2N\pi.$$

In order to show this we need some preliminary results. We will proceed as follows:

1. We will determine the general solution of the Euler-Lagrange equations (4.2.2) and (4.2.3).

2. We will then determine the constants in order that the above candidate satisfies the boundary conditions and is C^1 at $R = R_0$.

3. Then we will show that the constants are determined uniquely. In showing this an explicit form for the candidate for the minimiser is obtained.

In order to determine the solutions of the Euler-Lagrange equations (4.2.2) and (4.2.3) we put $t = \dfrac{R}{a}$, $T = \dfrac{b}{a}$, $\tilde{\rho}(t) = \dfrac{\rho(R)}{a}$ and $\tilde{\psi}(t) = \psi(R)$. Then $\tilde{\rho}'(t) = \rho'(R)$, $\tilde{\rho}''(t) = a\rho''(R)$ and $\tilde{\psi}'(t) = a\psi'(R)$. Hence the second Euler-Lagrange equation (4.2.3) becomes

$$t\tilde{\rho}^2(t)\tilde{\psi}'(t) = K, \tag{4.2.10}$$

where K is a constant of integration. Also the first Euler-Lagrange equation (4.2.2) becomes

$$t\tilde{\rho}''(t) + \tilde{\rho}'(t) - \frac{\tilde{\rho}(t)}{t} - t\tilde{\rho}(t)(\tilde{\psi}'(t))^2 = 0, \tag{4.2.11}$$

and substitution of (4.2.10) into (4.2.11) gives us that

$$t\tilde{\rho}''(t) + \tilde{\rho}'(t) - \frac{\tilde{\rho}(t)}{t} - \frac{K^2}{t\tilde{\rho}^3(t)} = 0. \tag{4.2.12}$$

We now show the following.

Lemma 4.2.2

$$\tilde{\rho}(t) = \frac{1}{\sqrt{2}}\left\{At^2 + \frac{B}{At^2} - C\right\}^{\frac{1}{2}}, \tag{4.2.13}$$

where

$$A = \frac{1}{T^2}\left\{T^2 + \frac{L}{2} + \left[LT^2 + T^4 - K^2\right]^{\frac{1}{2}}\right\}, \quad B = K^2 + \frac{L^2}{4} \quad \textit{and} \quad C = L, \tag{4.2.14}$$

satisfies (4.2.12) and the boundary condition $\tilde{\rho}(T) = T$, where K and L are constants of integration.

Proof: Upon making the change of variables $t = e^s$ and putting $\bar{\rho}(s) = \tilde{\rho}(t)$, if we multiply (4.2.12) by $\bar{\rho}'(s)$ and integrate with respect to s, (4.2.12) becomes

$$\left(\frac{d\bar{\rho}(s)}{ds}\right)^2 - \bar{\rho}^2(s) + \frac{K^2}{\bar{\rho}^2(s)} = L \tag{4.2.15}$$

where L is another constant of integration. In order to obtain an explicit form of $\bar{\rho}(s)$ satisfying (4.2.15) and $\bar{\rho}(\log T) = T$ where $T = \frac{b}{a}$, and hence a form of $\tilde{\rho}(t)$ satisfying (4.2.12) and $\tilde{\rho}(T) = T$, we can rewrite (4.2.15) as

$$\int_{\bar{\rho}(s)}^{T}\left(L + \frac{v^4 - K^2}{v^2}\right)^{-\frac{1}{2}} dv = \int_{\log t}^{\log T} ds = \log\left(\frac{T}{t}\right). \tag{4.2.16}$$

Putting $\bar{v} = v^2$ it can be shown from (4.2.16) that

$$\tilde{\rho}(t) = \frac{1}{\sqrt{2}}\left\{At^2 + \frac{B}{At^2} - C\right\}^{\frac{1}{2}}, \tag{4.2.17}$$

where

$$A = \frac{1}{T^2}\left\{T^2 + \frac{L}{2} + \left[LT^2 + T^4 - K^2\right]^{\frac{1}{2}}\right\}, \quad B = K^2 + \frac{L^2}{4} \quad \text{and} \quad C = L, \tag{4.2.18}$$

and $\tilde{\rho}(t)$ satisfies $\tilde{\rho}(T) = T$. $\square$

Hence, our candidate for $\tilde{\rho}(t)$ is

$$\tilde{\rho}(t) = \begin{cases} 1 & \text{on } [1, t_0] \\ \dfrac{1}{\sqrt{2}}\left\{At^2 + \dfrac{B}{At^2} - C\right\}^{\frac{1}{2}} & \text{on } [t_0, T] \end{cases} \tag{4.2.19}$$

since $\tilde{\rho}(1) = 1$ is a boundary condition, with A, B and C as given in (4.2.18). Note that $B > 0$ and $C^2 - 4B < 0$. (This will be useful in obtaining an expression for $\tilde{\psi}(t)$.)

We have yet to show that $\tilde{\rho}$ is C^1 at $t = t_0$ and that

$$\tilde{\rho}(t_0) = \frac{1}{\sqrt{2}}\left\{At_0^2 + \frac{B}{At_0^2} - C\right\}^{\frac{1}{2}} = 1, \tag{4.2.20}$$

$$\tilde{\rho}'(t_0) = \frac{\left(A - \dfrac{B}{At_0^3}\right)}{\sqrt{2}\left\{At_0^2 + \dfrac{B}{At_0^2} - C\right\}^{\frac{1}{2}}} = 0. \tag{4.2.21}$$

We now determine the constants A, B and C so that these conditions hold and, in particular, we want to show that these constants can be uniquely determined, so that we can say that there exists a solution where $\tilde{\rho}(t)$ of the form (4.2.19) satisfying the boundary conditions. We also wish to show that there is only one possible value of t_0 in $[a, b]$. In order to do this, we first show that A, B, C and t_0 can all be written in terms of T and K only.

Lemma 4.2.3 *Let $\tilde{\rho}(t)$ of the form (4.2.19) satisfy the conditions $\tilde{\rho}(T) = T$, $\tilde{\rho}'(t_0) = 0$ and $\tilde{\rho}(t_0) = 1$, and $\tilde{\psi}(t)$ satisfy (4.2.10). Then*

(1) the constants A, B, C and t_0 can all be written in terms of T and K;

(2) K must satisfy

$$2N\pi = K\ln(t_0) - \tan^{-1}\left(\frac{1}{K}\right) + \tan^{-1}\left(\frac{2AT^2 + 1 - K^2}{2K}\right). \tag{4.2.22}$$

Proof:

Part (1). From (4.2.21) we get that

$$t_0 = \left(\frac{B}{A^2}\right)^{\frac{1}{4}}. \tag{4.2.23}$$

Substitution of (4.2.23) into (4.2.20) gives that $2 = 2B^{\frac{1}{2}} - C$, which results in

$K^2 = 1 + L$. Hence $B = \dfrac{1}{4}(K^2+1)^2$, $C = K^2 - 1$ and

$$A = \frac{1}{T^2}\left\{T^2 + \frac{(K^2-1)}{2} + \left[(T^2-1)(T^2+K^2)\right]^{\frac{1}{2}}\right\}.$$

Hence the constants A, B and C can all be written in terms of T and the constant of integration K, and thus t_0 can be written in terms of T and K.

Part (2). From the Euler-Lagrange equation (4.2.10) we have that

$$\tilde{\psi}'(t) = \frac{K}{t\tilde{\rho}^2(t)} \tag{4.2.24}$$

and so as $\tilde{\psi}(1) = 0$ we have

$$2N\pi = \tilde{\psi}(T) = \int_{t=1}^{T} \frac{K}{t\tilde{\rho}^2(t)}\, dt, \tag{4.2.25}$$

which we can rearrange to get

$$K = \frac{2N\pi}{\left\{\displaystyle\int_{t=1}^{T} \frac{1}{t\tilde{\rho}^2(t)}\, dt\right\}}. \tag{4.2.26}$$

Substitution of $\tilde{\rho}(t)$ given by (4.2.19) into (4.2.26) gives that

$$K = \frac{2N\pi}{\left\{\displaystyle\int_{t=1}^{t_0} \frac{1}{t}\, dt + \int_{t=t_0}^{T} \frac{2}{t}\left(At^2 + \frac{B}{At^2} - C\right)^{-1} dt\right\}},$$

and so we have

$$\begin{aligned} 2N\pi &= K\ln(t_0) - \frac{2K}{(4B-C^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{2At_0^2 - C}{(4B-C^2)^{\frac{1}{2}}}\right) \\ &\qquad + \frac{2K}{(4B-C^2)^{\frac{1}{2}}}\tan^{-1}\left(\frac{2AT^2 - C}{(4B-C^2)^{\frac{1}{2}}}\right) \\ &= K\ln(t_0) - \tan^{-1}\left(\frac{2At_0^2 - C}{2K}\right) + \tan^{-1}\left(\frac{2AT^2 - C}{2K}\right) \\ &= K\ln(t_0) - \tan^{-1}\left(\frac{1}{K}\right) + \tan^{-1}\left(\frac{2AT^2 + 1 - K^2}{2K}\right) \end{aligned} \tag{4.2.27}$$

as required. □

We can also determine $\tilde{\psi}(t)$.

Lemma 4.2.4 *Let $\tilde{\psi}(t)$ be a solution of (4.2.24) satisfying $\tilde{\psi}(1) = 0$ and $\tilde{\psi}(T) = 2N\pi$. Then*

$$\tilde{\psi}(t) = \begin{cases} K\ln(t) & \text{on } [1, t_0] \\ \tan^{-1}\left(\dfrac{2At^2 - L}{2K}\right) - \tan^{-1}\left(\dfrac{2AT^2 - L}{2K}\right) + 2N\pi & \text{on } [t_0, T] \end{cases} \tag{4.2.28}$$

with $L = K^2 - 1$ and

$$A = \frac{1}{T^2}\left\{T^2 + \frac{K^2 - 1}{2} + \left[(K^2 - 1)T^2 + T^4 - K^2\right]^{\frac{1}{2}}\right\}.$$

Proof: We integrate (4.2.24) for the two cases where $t \in [1, t_0]$ and where $t \in [t_0, T]$. For $t \in [1, t_0]$ integration of (4.2.24) between 1 and t gives

$$\tilde{\psi}(t) = \int_{r=1}^{t} \frac{K}{r}\, dr = K\ln(t). \tag{4.2.29}$$

For $t \in [t_0, T]$ integration of (4.2.24) between t and T gives

$$\begin{aligned} 2N\pi - \tilde{\psi}(t) &= \int_{r=t}^{T} \frac{2}{r}\left(Ar^2 + \frac{B}{Ar^2} - C\right)^{-1} dr \\ &= \tan^{-1}\left(\frac{2AT^2 + 1 - K^2}{2K}\right) - \tan^{-1}\left(\frac{2At^2 + 1 - K^2}{2K}\right). \end{aligned}$$

Thus

$$\tilde{\psi}(t) = 2N\pi + \tan^{-1}\left(\frac{2At^2 + 1 - K^2}{2K}\right) - \tan^{-1}\left(\frac{2AT^2 + 1 - K^2}{2K}\right) \tag{4.2.30}$$

and thus $\tilde{\psi}(t)$ is given by (4.2.28) as required. □

Now recall that we must have $t_0 \in (1, T)$, that is

$$1 < \left(\frac{B}{A^2}\right)^{\frac{1}{4}} < T. \tag{4.2.31}$$

As we would like to show that for all $T > 1$ there exists a unique value of t_0 such that (4.2.27) holds, and as t_0 can be written in terms of T and K, it now remains to show that there is a unique value of K.

It can be shown that for (4.2.31) to hold either

$$-\infty < K < -\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}} \text{ or } \left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}} < K < \infty.$$

Let

$$\Omega(K) := \tan^{-1}\left(\frac{1}{K}\right) - \tan^{-1}\left(\frac{2AT^2+1-K^2}{2K}\right) + 2N\pi - K\ln(t_0) \quad (4.2.32)$$

with t_0 given by (4.2.23). We want $\Omega(K) = 0$. Note that $\Omega(K) = \tilde{\psi}(t_0^+) - \tilde{\psi}(t_0^-)$.

Lemma 4.2.5 *There is a unique value of K in the region*

$$\left(-\infty, -\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}\right) \cup \left(\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}, \infty\right) \quad (4.2.33)$$

satisfying $\Omega(K) = 0$.

Proof: We proceed in the following way:

(a) We differentiate $\Omega(K)$ with respect to K, with a view to showing that for K in the region (4.2.33) $\Omega'(K)$ is one-signed. If this is the case then there must be at most one value of K in each of the two regions

$$\left(-\infty, -\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}\right) \text{ and } \left(\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}, \infty\right)$$

such that $\Omega(K) = 0$.

(b) We then consider the behaviour of $\Omega(K)$ as $K \to \pm\infty$ and as

$$K \to \pm\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}.$$

Here we will show that $\Omega\left(-\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}\right)$ and $\lim_{K\to-\infty}\Omega(K)$ are both positive, and also that $\Omega\left(\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}}\right)$ is positive but $\lim_{K\to\infty}\Omega(K)$ is negative, and thus that there is only one value of K such that K is in (4.2.33) and $\Omega(K) = 0$.

(See Figure 4.7.)

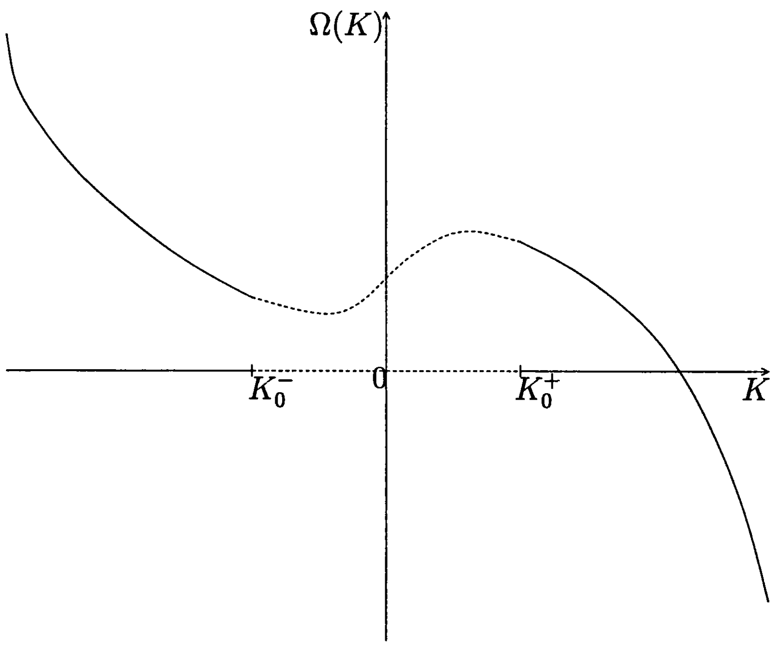

Figure 4.7: A possible graph for $\Omega(K)$ for K in the region (4.2.33).

For (a), since $\Omega(K)$ is a very complicated-looking expression, the computer algebra system Maple V can be used to help establish the sign of $\Omega'(K)$ in the required region. It can be shown, upon using Maple V, putting

$$K = \pm\left(\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}} + n\right)$$

where $n > 0$, and after some very involved algebraic manipulation, that $\Omega'(K) < 0$ for all possible $T > 1$, and thus $\Omega'(K) < 0$ when K lies in the region given in (4.2.33).

For (b), it can be seen that as $K \to \infty$, $\Omega(K) \to -\infty$ and as $K \to -\infty$, $\Omega(K) \to \infty$. Also

$$K = \pm\left(\frac{1+3T^2}{T^2-1}\right)^{\frac{1}{2}} \tag{4.2.34}$$

corresponds to $t_0 = 1$. Hence putting K_0^+ and K_0^- to be the two values of K as given by (4.2.34) we see that

$$\Omega(K_0^+) = \tan^{-1}\left(\frac{2A+1-(K_0^+)^2}{2K_0^+}\right) - \tan^{-1}\left(\frac{2AT^2+1-(K_0^+)^2}{2K_0^+}\right) + 2N\pi$$

and

$$\Omega(K_0^-) = \tan^{-1}\left(\frac{2A+1-(K_0^-)^2}{2K_0^-}\right) - \tan^{-1}\left(\frac{2AT^2+1-(K_0^-)^2}{2K_0^-}\right) + 2N\pi.$$

Putting

$$V = \tan^{-1}\left(\frac{2A+1-(K_0^+)^2}{2K_0^+}\right) - \tan^{-1}\left(\frac{2AT^2+1-(K_0^+)^2}{2K_0^+}\right)$$

we see that $\Omega(K_0^+) = 2N\pi + V$ and $\Omega(K_0^+) = 2N\pi - V$. If we assume that

$$-\frac{\pi}{2} \leq \tan^{-1}\left(\frac{2A+1-(K_0^+)^2}{2K_0^+}\right) \leq \frac{\pi}{2}$$

and

$$-\frac{\pi}{2} \leq \tan^{-1}\left(\frac{2AT^2+1-(K_0^+)^2}{2K_0^+}\right) \leq \frac{\pi}{2}$$

then $-\pi \leq V \leq \pi$ and hence $(2N-1)\pi \leq \Omega(K_0^+) \leq (2N+1)\pi$ and $(2N-1)\pi \leq \Omega(K_0^-) \leq (2N+1)\pi$. In any case we have that $\Omega(K_0^+) > 0$ and $\Omega(K_0^-) > 0$. Hence as $\Omega(K_0^-) > 0$, $\Omega(K) \to \infty$ as $K \to -\infty$ and $\Omega'(K) < 0$ for $K \in (-\infty, K_0^-)$ it follows that there are no values of K in $(-\infty, K_0^-)$ such that $\Omega(K) = 0$. Also as $\Omega(K_0^+) > 0$, $\Omega(K) \to -\infty$ as $K \to \infty$ and $\Omega'(K) < 0$ for $K \in (K_0^+, \infty)$ it follows that there is only one value of K in (K_0^+, ∞) such that $\Omega(K) = 0$. Hence there is a unique K such that $\Omega(K) = 0$, given that $t_0 \in (1, T)$. □

We are now in a position to prove Proposition 4.2.1.

Proof of Proposition 4.2.1: By Lemma 4.2.3 and Lemma 4.2.5, the constants A, B and C in $\tilde{\rho}(t)$ given by (4.2.19) can be uniquely determined, and so by Lemma 4.2.2 we can write down a candidate for $\tilde{\rho}(t)$ satisfying the necessary conditions.

By Lemma 4.2.4, a possible candidate for $\tilde{\psi}(t)$ is

$$\tilde{\psi}(t) = \begin{cases} K\ln(t) & \text{on } [1, t_0] \\ \tan^{-1}\left(\dfrac{2At^2 - L}{2K}\right) - \tan^{-1}\left(\dfrac{2AT^2 - L}{2K}\right) + 2N\pi & \text{on } [t_0, T] \end{cases} \tag{4.2.35}$$

with K being uniquely determined, $L = K^2 - 1$ and

$$A = \frac{1}{T^2}\left\{T^2 + \frac{K^2-1}{2} + \left[(K^2-1)T^2 + T^4 - K^2\right]^{\frac{1}{2}}\right\}.$$

Hence our candidate for $\rho(R)$ and $\psi(R)$ are

$$\rho(R) = \begin{cases} a & \text{on } [a, R_0] \\ \dfrac{1}{\sqrt{2}}\left\{\hat{A}R^2 + \dfrac{\hat{B}}{\hat{A}R^2} - \hat{C}\right\}^{\frac{1}{2}} & \text{on } [R_0, b] \end{cases} \tag{4.2.36}$$

and

$$\psi(R) = \begin{cases} \dfrac{k}{a^2}\log\left(\dfrac{R}{a}\right) & \text{on } [a, R_0] \\ \tan^{-1}\left(\dfrac{2\hat{A}R^2 - \hat{C}}{2k}\right) - \tan^{-1}\left(\dfrac{2\hat{A}b^2 - \hat{C}}{2k}\right) + 2N\pi & \text{on } [R_0, b], \end{cases} \tag{4.2.37}$$

where $k = a^2K$,

$$\hat{A} = \frac{1}{b^2}\left\{b^2 + \frac{1}{2}\left(\frac{k^2}{a^2}\right) + \left[\frac{1}{a^2}(b^2 - a^2)(a^2b^2 + k^2)\right]^{\frac{1}{2}}\right\}, \quad \hat{B} = \frac{1}{4}\left(a^2 + \frac{k^2}{a^2}\right)^2$$

and $\hat{C} = a^2L = \dfrac{k^2}{a^2} - a^2$, and where R_0 is such that $\rho'(R_0) = 0$, $\rho(R_0) = a$ and $\rho(R) > 0$ for $R > R_0$. □

This is our candidate for the minimiser. As already noted, it will be shown in Chapter 6 that this candidate, which is rotationally symmetric, minimises the energy functional in a more general class of deformations.

Chapter 5

Rotationally symmetric deformations of an incompressible annulus.

In this chapter we consider rotationally symmetric deformations of the form (4.0.5) of an incompressible annulus in which the deformations are subject to the additional constraint

$$\det(\nabla \boldsymbol{u}) = 1 \text{ a.e. in } A. \tag{5.0.1}$$

Thus we are considering a more restrictive class of deformations. Recall that for deformations of the form (4.0.5) we have (4.0.8). Hence, $\det(\nabla \boldsymbol{u}) = \rho' \frac{\rho}{R} = 1$ from (5.0.1), and so $\rho(R) = (R^2 + c)^{\frac{1}{2}}$. The boundary conditions $\rho(a) = a$, $\rho(b) = b$ result in $c = 0$, and so for incompressible deformations $\boldsymbol{u}$ of the form (4.0.5) we must have

$$\rho(R) \equiv R \text{ a.e. in } [a, b]. \tag{5.0.2}$$

Thus from (4.0.8) we have $|\nabla \boldsymbol{u}|^2 = 2 + (R\psi')^2$.
We will consider polyconvex stored energy functions of the form $W(\nabla \boldsymbol{u}) = \bar{g}(|\nabla \boldsymbol{u}|)$ and, as before, if we also assume that $\bar{g}'(|F|) > 0$ then $\bar{g}$ is convex in ψ'. Hence we now wish to show that there exist minimisers for $E(\boldsymbol{u})$ given by

$$E(\boldsymbol{u}) = 2\pi I(\psi) := 2\pi \int_a^b R\bar{g}(\{2 + (R\psi')^2\}^{\frac{1}{2}})dR \tag{5.0.3}$$

on the set

$$\mathcal{A}_N^{\rm inc} = \{\psi \in W^{1,1}((a,b)) : \psi(a) = 0,\ \psi(b) = 2N\pi\}. \tag{5.0.4}$$

This is achieved by means of the following result.

Proposition 5.0.1 *Let W be of the form $W(F) = \bar{g}(|F|)$. Let $\bar{g} : \mathbb{R} \to \mathbb{R}$ be a C^2 convex function such that $\bar{g}'(|F|) \geq 0$ and*

$$\bar{g}(|F|) \geq C_1|F|^p + C_2$$

for all $F \in M_1^{2\times 2}$ where $p \geq 2$; $C_1 > 0$ and C_2 are constants. Then there exists a minimiser for $I(\psi)$ on $\mathcal{A}_N^{\rm inc}$ for each $N \in \mathbb{N} \cup \{0\}$, where $I(\psi)$ is given by (5.0.3) and $\mathcal{A}_N^{\rm inc}$ is given by (5.0.4).
Further, if $\bar{g}$ is a strictly convex function and $\bar{\psi}$ is a minimiser of $I(\psi)$ in $\mathcal{A}_N^{\rm inc}$ then $\bar{\psi} \in C^2((a,b))$ and $\bar{\psi}$ satisfies the equation

$$\frac{d}{dR}\left[\frac{R^3\psi'}{S}\bar{g}'(S)\right] = 0 \tag{5.0.5}$$

for $R \in [a,b]$, where $S = |\nabla \boldsymbol{u}| = \{1 + 1 + (R\psi')^2\}^{\frac{1}{2}}$.

The method of proving this result is analogous to that of Propositions 4.1.2 & 4.1.3 and is omitted.
The next result is one in which we show the relationship between solutions $\bar{\psi}$ of (5.0.5) and solutions of the equilibrium equations for E. In order to show this we will consider the weak form of the equilibrium equations for an incompressible body. As noted in [8], the equilibrium equations for an incompressible two-dimensional body, that is

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F_\alpha^i}(\nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\mathrm{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))_i^\alpha\right] = 0,\ \boldsymbol{x} \in A,\ i = 1,2, \tag{5.0.6}$$

are the Euler-Lagrange equations for the functional

$$E(\boldsymbol{u}) = \int_A W(\nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})\{\det(\nabla \boldsymbol{u}(\boldsymbol{x}))) - 1\}\ \mathrm{dx},$$

and thus in the proof of the following result we will consider the weak form of the equilibrium equations, that is

$$\int_A \left\{\frac{\partial W}{\partial F}(\nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\mathrm{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))^T\right\} \cdot \nabla \boldsymbol{\phi}(\boldsymbol{x})\ \mathrm{dx} = 0 \tag{5.0.7}$$

for $\boldsymbol{\phi} \in C_0^1(A)$.

Proposition 5.0.2 *Let $W(F)$ be of the form $W(F) = \bar{g}(|F|)$. If ψ is a solution of*

$$\frac{d}{dR}\left[\frac{R^3\psi'}{S}\bar{g}'(S)\right] = 0 \tag{5.0.8}$$

where $S = |\nabla \boldsymbol{u}| = \{2+(R\psi')^2\}^{\frac{1}{2}}$, then there exists a hydrostatic pressure $\bar{P}(\boldsymbol{x}) = P(R)$ (unique up to a constant) where

$$\frac{\partial P}{\partial R}(R) = \frac{1}{R}\frac{d}{dR}\left[\frac{R}{S}\bar{g}'(S)\right] - \frac{\bar{g}'(S)}{RS} - \frac{R(\psi')^2}{S}\bar{g}'(S)$$

such that a solution ψ of (5.0.8) gives rise to a corresponding solution $\boldsymbol{u}$ of the full Euler-Lagrange equations for E, that is (5.0.6).

Proof: We will first show that (5.0.7) is satisfied for all $\boldsymbol{\phi} \in C_0^1(A)$, since if this is the case then as $\psi \in C^2((a,b))$ it follows that (5.0.6) holds. As $W(F) = \bar{g}(|F|)$ we have

$$\frac{\partial W}{\partial F}(F) = \frac{\bar{g}'(|F|)}{|F|}F.$$

Now putting $W(\nabla \boldsymbol{u}) = \bar{W}\left(\rho', \frac{\rho}{R}, \rho\psi'\right)$, we have $\frac{\partial W}{\partial F}$ such that the set of equations (4.1.38) holds.
Also we consider variations $\boldsymbol{\phi}(\boldsymbol{x})$ of the form (4.1.40) satisfying the same conditions as in the proof of Proposition 4.1.5, except that now $\psi(R)$ is such that $I(\psi) = \inf_{\mathcal{A}_N^{\text{inc}}} I$ and obtain $\nabla\boldsymbol{\phi}$ to be of the form (4.1.41).
Now for this case we have that

$$\frac{\partial \bar{W}}{\partial(\rho')} = \frac{\rho'}{S}\bar{g}'(S), \quad \frac{\partial \bar{W}}{\partial \rho} = \frac{\bar{g}'(S)}{S}\left\{\frac{\rho}{R^2} + \rho(\psi')^2\right\} = \frac{1}{R}\frac{\partial \bar{W}}{\partial\left(\frac{\rho}{R}\right)}$$
$$\frac{\partial \bar{W}}{\partial(\rho\psi')} = \frac{\rho\psi'}{S}\bar{g}'(S) = \frac{1}{\rho}\frac{\partial \bar{W}}{\partial \psi'}$$

where $S = \left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}$.

Also

$$(\mathrm{adj}(\nabla \boldsymbol{u}))^T.\nabla \boldsymbol{\phi} = \mathrm{tr}((\mathrm{adj}(\nabla \boldsymbol{u}))^T (\nabla \boldsymbol{\phi})^T) =$$

$$\mathrm{tr}\left(\begin{pmatrix} \dfrac{\rho}{R} & -\rho\psi' \\ 0 & \rho' \end{pmatrix} \begin{pmatrix} \dfrac{\partial \phi_1}{\partial R} & \phi_1\psi' \\ \dfrac{1}{R}\dfrac{\partial \phi_1}{\partial \theta} & \dfrac{\phi_1}{R} \end{pmatrix} + \begin{pmatrix} 0 & -\rho' \\ -\dfrac{\rho}{R} & \rho\psi' \end{pmatrix} \begin{pmatrix} \dfrac{\partial \phi_2}{\partial R} & \phi_2\psi' \\ \dfrac{1}{R}\dfrac{\partial \phi_2}{\partial \theta} & \dfrac{\phi_2}{R} \end{pmatrix} \right)$$

$$= \frac{\rho}{R}\left\{ \frac{\partial \phi_1}{\partial R} - \phi_2\psi' \right\} + \rho\psi' \left\{ \frac{1}{R}\frac{\partial \phi_1}{\partial \theta} + \frac{\phi_2}{R} \right\} + \rho' \left\{ \frac{\phi_1}{R} - \frac{1}{R}\frac{\partial \phi_2}{\partial \theta} \right\}. \tag{5.0.9}$$

Hence we have

$$\int_A \left\{ \frac{\partial W}{\partial F}(\nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\mathrm{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))^T \right\}.\nabla \boldsymbol{\phi}(\boldsymbol{x})\ \mathrm{dx} =$$

$$\int_{R=a}^{b} \int_{\theta=0}^{2\pi} \left\{ \frac{R\rho'}{S}\bar{g}'(S) - \hat{P}(R,\theta)\rho \right\} \frac{\partial \phi_1}{\partial R} + \left\{ \left[\frac{\rho}{R} + R\rho(\psi')^2 \right] \frac{\bar{g}'(S)}{S} - \hat{P}(R,\theta)\rho' \right\} \phi_1$$

$$+ \frac{R\rho\psi'}{S}\bar{g}'(S)\frac{\partial \phi_2}{\partial R} - R\rho'\psi'\frac{\bar{g}'(S)}{S}\phi_2 + \frac{\partial \phi_2}{\partial \theta}\rho'\hat{P}(R,\theta) - \frac{\partial \phi_1}{\partial \theta}\rho\psi'\hat{P}(R,\theta)\ dR\ d\theta \tag{5.0.10}$$

and as $\rho(R) \equiv R$ we have

$$\int_A \left\{ \frac{\partial W}{\partial F}(\nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\mathrm{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))^T \right\}.\nabla \boldsymbol{\phi}(\boldsymbol{x})\ \mathrm{dx} =$$

$$\int_{R=a}^{b} \int_{\theta=0}^{2\pi} R\left\{ \frac{\bar{g}'(S)}{S} - \hat{P}(R,\theta) \right\} \frac{\partial \phi_1}{\partial R} + \left\{ [1 + R^2(\psi')^2] \frac{\bar{g}'(S)}{S} - \hat{P}(R,\theta) \right\} \phi_1$$

$$+ \frac{R^2\psi'}{S}\bar{g}'(S)\frac{\partial \phi_2}{\partial R} - R\psi'\frac{\bar{g}'(S)}{S}\phi_2 + \frac{\partial \phi_2}{\partial \theta}\hat{P}(R,\theta) - R\frac{\partial \phi_1}{\partial \theta}\psi'\hat{P}(R,\theta)\ dR\ d\theta \tag{5.0.11}$$

and S is now $\{2 + (R\psi')^2\}^{\frac{1}{2}}$.
Let $\psi(R)$ be a solution of the Euler-Lagrange equation

$$\frac{d}{dR}\left[\frac{R^3\psi'}{S}\bar{g}'(S) \right] = 0.$$

Then we can say that

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi}\left\{\frac{1}{R}\frac{d}{dR}\left[\frac{R^3\psi'}{S}\bar{g}'(S)\right]\right\}\phi_2\,dR\,d\theta = 0.$$

Hence the first variation is now

$$\begin{aligned}\int_A\left\{\frac{\partial W}{\partial F}(\nabla\boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\mathrm{adj}(\nabla\boldsymbol{u}(\boldsymbol{x})))^T\right\}\nabla\boldsymbol{\phi}(\boldsymbol{x})\ \mathrm{dx} = \\ \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{\frac{\bar{g}'(S)}{S} - \hat{P}(R,\theta)\right\}\frac{\partial\phi_1}{\partial R} + \left\{\left[1+R^2(\psi')^2\right]\frac{\bar{g}'(S)}{S} - \hat{P}(R,\theta)\right\}\phi_1 \\ + \frac{\partial\phi_2}{\partial\theta}\hat{P}(R,\theta) - R\frac{\partial\phi_1}{\partial\theta}\psi'\hat{P}(R,\theta)\ dR\,d\theta.\end{aligned} \tag{5.0.12}$$

Now we can choose $\hat{P}(R,\theta) = P(R)$ where

$$\frac{dP}{dR}(R) = \frac{1}{R}\frac{d}{dR}\left[\frac{R}{S}\bar{g}'(S)\right] - \frac{\bar{g}'(S)}{RS} - \frac{R(\psi')^2}{S}\bar{g}'(S).$$

Then, upon integration by parts of (5.0.12), it can be shown that (5.0.7) holds. As $\psi(R) \in C^2((a,b))$ it can further be shown that (5.0.6) holds as required. □
We note that the stored energy function defined in (5.0.3) is strictly convex in ψ' and hence for each $N \in \mathbb{N}$ there exists at most one smooth solution $\bar{\psi}$ of the Euler-Lagrange equation satisfying the corresponding boundary conditions.

Example 5.0.3 *We now consider the case where the stored energy function is $W(F) = \frac{1}{2}|F|^2$. Thus the total energy (5.0.3) takes the form*

$$E(\boldsymbol{u}) = \pi\int_a^b R(2 + (R\psi')^2)\ dR.$$

This satisfies the conditions of Propositions 5.0.1 and 5.0.2. Thus $\bar{\psi}$ satisfies

$$\frac{d}{dR}[R^3\psi'(R)] = 0. \tag{5.0.13}$$

In this case the associated hydrostatic pressure term $P(R)$ is such that

$$P'(R) = -R(\bar{\psi}'(R))^2. \tag{5.0.14}$$

Hence using the boundary conditions $\psi(a) = 0$, $\psi(b) = 2N\pi$ we find that

$$\bar{\psi}(R) = \frac{2N\pi a^2 b^2}{b^2 - a^2}\left(\frac{1}{a^2} - \frac{1}{R^2}\right) \text{ and } P(R) = \frac{4N^2\pi^2 a^4 b^4}{(b^2 - a^2)^2 R^4} \tag{5.0.15}$$

satisfy (5.0.13) and (5.0.14). (Note that $\bar{\psi}(R)$ is monotonic.)

It will be shown in Chapter 6 that the rotationally symmetric incompressible equilibrium solution constructed in the above example is a weak local minimiser in a general class of deformations.

Chapter 6

Minimising properties of rotationally symmetric equilibria.

In this chapter we will consider the minimisation properties of the explicit rotationally symmetric equilibrium solutions obtained for the energy functional

$$E(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}|^2 \, \mathrm{dx} \tag{6.0.1}$$

in both the degenerate case (§4.2) and the incompressible case (Example 5.0.3) with respect to more general, not necessarily rotationally symmetric deformations.

We will show in §6.1 (the degenerate compressible case) that the rotationally symmetric candidate constructed in §4.2 minimises the energy functional in a class of non-rotationally symmetric admissible variations, and hence rotationally symmetric equilibria are minimisers in this class of admissible deformations. We will also show in §6.2 (the incompressible case) that the rotationally symmetric equilibria obtained in Example 5.0.3 are weak local minimisers in a general class of admissible incompressible deformations. In §6.3 we will turn our attention to extending the analysis of §6.2 to the case of deformations of an annulus composed of slightly compressible elastic material, where the energy functional $E_\delta(\boldsymbol{u})$ is that for an annulus composed of compressible neo-Hookean material and is of the form

$$E_\delta(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}|^2 + \frac{1}{\delta} h(\det(\nabla \boldsymbol{u})) \, \mathrm{dx}. \tag{6.0.2}$$

6.1 The degenerate case: minimisation of the Dirichlet integral.

Recall that in §4.2 the candidate for $\rho(R)$ and $\psi(R)$ that we obtained was given by

$$\rho(R) = \begin{cases} a & \text{on } [a, R_0] \\ \dfrac{1}{\sqrt{2}}\left\{\hat{A}R^2 + \dfrac{\hat{B}}{\hat{A}R^2} - \hat{C}\right\}^{\frac{1}{2}} & \text{on } [R_0, b] \end{cases} \tag{6.1.1}$$

and

$$\psi(R) = \begin{cases} \dfrac{k}{a^2}\log\left(\dfrac{R}{a}\right) & \text{on } [a, R_0] \\ \tan^{-1}\left(\dfrac{2\hat{A}R^2 - \hat{C}^2}{2k}\right) - \tan^{-1}\left(\dfrac{2\hat{A}b^2 - \hat{C}^2}{2k}\right) + 2N\pi & \text{on } [R_0, b], \end{cases} \tag{6.1.2}$$

with

$$\hat{A} = \frac{1}{b^2}\left\{b^2 + \frac{1}{2}\left(\frac{k^2}{a^2}\right) + \left[\frac{1}{a^2}(b^2 - a^2)(a^2b^2 + k^2)\right]^{\frac{1}{2}}\right\}, \quad \hat{B} = \frac{1}{4}\left(a^2 + \frac{k^2}{a^2}\right)^2$$

and $\hat{C} = \dfrac{k^2}{a^2} - a^2$, and where R_0 was such that $\rho'(R_0) = 0$, $\rho(R_0) = a$ and $\rho'(R) > 0$ for $R > R_0$. We now want to show that this choice of (ρ, ψ) minimises E given by (6.0.1). The approach we take is based on a method used in [64]. Consider the map $\hat{\boldsymbol{u}}(\boldsymbol{x}) + \epsilon\boldsymbol{\phi}(\boldsymbol{x})$ where $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is of the form

$$\hat{\boldsymbol{u}}(\boldsymbol{x}) = \rho(R)\begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix}, \tag{6.1.3}$$

where $\rho(R)$ given by (6.1.1) and $\psi(R)$ given by (6.1.2) are our candidate for the minimiser. We first introduce the concept of an admissible variation.

Definition 6.1.1 ([64, Definition 2.1] page 671) *Given any deformation $\boldsymbol{u}$ of A, we say that $\boldsymbol{\phi} \in W_0^{1,p}(A)$, $p \geq 2$, is an* ***admissible variation*** *of $\boldsymbol{u}$ if for almost all $\boldsymbol{x} \in A$ and for all $\epsilon > 0$ sufficiently small we have* $\det(\nabla\boldsymbol{u}(\boldsymbol{x}) + \epsilon\nabla\boldsymbol{\phi}(\boldsymbol{x})) \geq 0$ *and $\boldsymbol{u} + \epsilon\boldsymbol{\phi} : \bar{A} \to \bar{A}$.*

The reason why we have to consider admissible variations is due to the fact that we do not want $\det(\nabla \boldsymbol{u}(\boldsymbol{x}) + \epsilon \nabla \boldsymbol{\phi}(\boldsymbol{x})) < 0$ on a set of positive measure and so as we are considering deformations where $\det(\nabla \boldsymbol{u}) \geq 0$ we are not allowed to consider two-sided variations around an equilibrium solution throughout A.
We write $\boldsymbol{\phi}(\boldsymbol{x})$ in the form

$$\boldsymbol{\phi}(\boldsymbol{x}) = \Pi(R,\theta)\begin{pmatrix}\cos(\theta+\psi(R)) \\ \sin(\theta+\psi(R))\end{pmatrix} + \Lambda(R,\theta)\begin{pmatrix}-\sin(\theta+\psi(R)) \\ \cos(\theta+\psi(R))\end{pmatrix}, \qquad (6.1.4)$$

where $\psi(R)$ is as in (6.1.2). We now determine the conditions on $\Pi(R,\theta)$ and $\Lambda(R,\theta)$ such that $\boldsymbol{\phi}(\boldsymbol{x})$ is an admissible variation.

Proposition 6.1.2 *For $\boldsymbol{\phi}$ of the form (6.1.4) to be an admissible variation of $\hat{\boldsymbol{u}}$ given by (6.1.3) we must have*

(A1) $\Pi \geq 0$,

(A2) $\dfrac{\partial \Pi}{\partial R} - \dfrac{k}{a^2 R}\dfrac{\partial \Pi}{\partial \theta} \geq 0$

for $R \in [a, R_0]$. (Note that there is no restriction on Λ.)

Proof: If we write

$$\nabla \hat{\boldsymbol{u}} = \begin{pmatrix}\hat{u}^1_{,1} & \hat{u}^1_{,2} \\ \hat{u}^2_{,1} & \hat{u}^2_{,2}\end{pmatrix} \qquad \text{and} \qquad \nabla \boldsymbol{\phi} = \begin{pmatrix}\phi^1_{,1} & \phi^1_{,2} \\ \phi^2_{,1} & \phi^2_{,2}\end{pmatrix}$$

we require that $(\hat{u}^1_{,1} + \epsilon\phi^1_{,1})(\hat{u}^2_{,2} + \epsilon\phi^2_{,2}) - (\hat{u}^1_{,2} + \epsilon\phi^1_{,2})(\hat{u}^2_{,1} + \epsilon\phi^2_{,1}) \geq 0$, and hence that $\hat{u}^1_{,1}\hat{u}^2_{,2} - \hat{u}^1_{,2}\hat{u}^2_{,1} + \epsilon(\hat{u}^1_{,1}\phi^2_{,2} + \phi^1_{,1}\hat{u}^2_{,2} - \hat{u}^1_{,2}\phi^2_{,1} - \hat{u}^2_{,1}\phi^1_{,2}) + \epsilon^2(\phi^1_{,1}\phi^2_{,2} - \phi^1_{,2}\phi^2_{,1}) \geq 0$. Now $\det(\nabla\hat{\boldsymbol{u}}) = \hat{u}^1_{,1}\hat{u}^2_{,2} - \hat{u}^1_{,2}\hat{u}^2_{,1} = 0$ on $B_{R_0} \setminus B_a$ since $\rho'(R) = 0$ for $R \in [a, R_0]$, and so for $\epsilon \geq 0$ sufficiently small we require $\hat{u}^1_{,1}\phi^2_{,2} + \phi^1_{,1}\hat{u}^2_{,2} - \hat{u}^1_{,2}\phi^2_{,1} - \hat{u}^2_{,1}\phi^1_{,2} \geq 0$. It can be shown that

$$\hat{u}^1_{,1}\phi^2_{,2} + \phi^1_{,1}\hat{u}^2_{,2} - \hat{u}^1_{,2}\phi^2_{,1} - \hat{u}^2_{,1}\phi^1_{,2} = \frac{a}{R^3}\left(-\frac{\partial\psi}{\partial R}\frac{\partial\Pi}{\partial\theta} + \frac{\partial\Pi}{\partial R}\right)$$

and so we require

$$\frac{\partial\Pi}{\partial R} - \frac{\partial\Pi}{\partial\theta}\frac{\partial\psi}{\partial R} \geq 0 \qquad \text{for } R \in [a, R_0],$$

that is

$$\frac{\partial\Pi}{\partial R} - \frac{k}{a^2 R}\frac{\partial\Pi}{\partial\theta} \geq 0 \qquad \text{for } R \in [a, R_0], \qquad (6.1.5)$$

for $\boldsymbol{\phi}$ to be an admissible variation.
Now we consider $\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi}$. We require $\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi} : \bar{A} \to \bar{A}$ for $\epsilon > 0$ sufficiently small. For this to be true we must have $|\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi}| \geq a$. Now, $|\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi}|^2 = (a+\epsilon\Pi)^2+\epsilon^2\Lambda^2$, and so we want $(a+\epsilon\Pi)^2+\epsilon^2\Lambda^2 \geq a^2$, and so we require $\Pi \geq 0$ for R on $[a, R_0]$ in order that $\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi} : \bar{A} \to \bar{A}$ for $\epsilon \geq 0$ sufficiently small. □

Remark 6.1.3 *It is possible to have variations $\boldsymbol{\phi}$ of the form (6.1.4) such that* (A1) *holds but* (A2) *does not necessarily hold throughout $[a, R_0]$, one example being* $\Pi(R,\theta) = \dfrac{1}{R-(a-\delta)}\sin\left(\pi\dfrac{R-a}{b-a}\right)$, $\Lambda(R,\theta) \equiv 0$, *where* $0 < \delta \ll 1$.

The main result in this section is the following.

Proposition 6.1.4 *Let $\hat{\boldsymbol{u}}(\boldsymbol{x})$ be the rotationally symmetric deformation defined by (6.1.3) where $\rho(R)$ is defined by (6.1.1) and $\psi(R)$ is defined by (6.1.2). Then*

$$E(\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi}) \geq E(\hat{\boldsymbol{u}}) + \frac{\epsilon^2}{2}\int_A |\nabla\boldsymbol{\phi}|^2 \mathrm{dx}$$

for all admissible variations $\boldsymbol{\phi} \in W_0^{1,2}(A)$.

Proof: For deformations of the form $\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi}$ the energy is given by

$$\begin{aligned} E(\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi}) &= \int_A \frac{1}{2}|\nabla(\hat{\boldsymbol{u}}+\epsilon\boldsymbol{\phi})|^2 \,\mathrm{dx} \\ &= \int_A \frac{1}{2}|\nabla\hat{\boldsymbol{u}}|^2 \,\mathrm{dx} + \epsilon^2\int_A \frac{1}{2}|\nabla\boldsymbol{\phi}|^2 \,\mathrm{dx} + \epsilon\int_A \nabla\hat{\boldsymbol{u}}.\nabla\boldsymbol{\phi} \,\mathrm{dx} \end{aligned}$$

and hence we require to show that

$$\int_A \nabla\hat{\boldsymbol{u}}.\nabla\boldsymbol{\phi} \,\mathrm{dx} \geq 0 \tag{6.1.6}$$

for all admissible $\boldsymbol{\phi} \neq \mathbf{0}$. The proof proceeds in two stages:
Step 1. Using index notation, where

$$v^i_{,\alpha} = \frac{\partial v^i}{\partial x_\alpha},$$

we can write

$$\begin{aligned} \int_A \nabla\hat{\boldsymbol{u}}.\nabla\boldsymbol{\phi} \,\mathrm{dx} &= \int_A \hat{u}^i_{,\alpha}\phi^i_{,\alpha} \,\mathrm{dx} \\ &= \int_A (\hat{u}^i_{,\alpha}\phi^i)_{,\alpha} - \phi^i\hat{u}^i_{,\alpha\alpha} \,\mathrm{dx} \\ &= \int_{B_b\backslash B_{R_0}} (\hat{u}^i_{,\alpha}\phi^i)_{,\alpha} - \phi^i\hat{u}^i_{,\alpha\alpha} \,\mathrm{dx} + \int_{B_{R_0}\backslash B_a} (\hat{u}^i_{,\alpha}\phi^i)_{,\alpha} - \phi^i\hat{u}^i_{,\alpha\alpha} \,\mathrm{dx} \end{aligned}$$

(as $A = B_b \setminus B_a = B_b \setminus B_{R_0} \cup B_{R_0} \setminus B_a$ where B_c denotes a disc of radius c)

$$\begin{aligned} &= \int_{\partial B_b} \phi^i \hat{u}^i_{,\alpha} n^\alpha \text{ ds} - \int_{\partial B_{R_0}} \phi^i \hat{u}^i_{,\alpha} n^\alpha \text{ ds} - \int_{B_b \setminus B_{R_0}} \phi^i \hat{u}^i_{,\alpha\alpha} \text{ dx} \\ &+ \int_{\partial B_{R_0}} \phi^i \hat{u}^i_{,\alpha} n^\alpha \text{ ds} - \int_{\partial B_a} \phi^i \hat{u}^i_{,\alpha} n^\alpha \text{ ds} - \int_{B_{R_0} \setminus B_a} \phi^i \hat{u}^i_{,\alpha\alpha} \text{ dx} \end{aligned} \tag{6.1.7}$$

by use of the divergence theorem (where $\boldsymbol{n} = (n^\alpha)$ is the outward unit normal). Now

$$\int_{B_b \setminus B_{R_0}} \phi^i \hat{u}^i_{,\alpha\alpha} \text{ dx} = 0 \tag{6.1.8}$$

since in this region $\hat{\boldsymbol{u}}$ satisfies the Euler-Lagrange equation $\Delta \hat{\boldsymbol{u}} = \mathbf{0}$.
Also, as the outward unit normal to the region $B_b \setminus B_{R_0}$ on ∂B_{R_0} is in the opposite direction to that to the region $B_{R_0} \setminus B_a$ on ∂B_{R_0} and $\hat{u}$ is C^1 across ∂B_{R_0} (since $\rho(R)$ and $\psi(R)$ are C^1 at R_0) we can say that

$$\int_{\partial B_b \cup \partial B_{R_0}} \phi^i \hat{u}^i_{,\alpha} n^\alpha \text{ ds} + \int_{\partial B_{R_0} \cup \partial B_a} \phi^i \hat{u}^i_{,\alpha} n^\alpha \text{ ds} = 0, \tag{6.1.9}$$

since $\phi^i = 0$ on ∂A. Hence putting (6.1.8) and (6.1.9) into (6.1.7) we have that

$$\int_A \nabla \hat{\boldsymbol{u}} . \nabla \boldsymbol{\phi} \text{ dx} = - \int_{B_{R_0} \setminus B_a} \boldsymbol{\phi} . \Delta \hat{\boldsymbol{u}} \text{ dx}.$$

Hence to prove (6.1.6) it is sufficient to show that $-\boldsymbol{\phi}.\Delta \hat{\boldsymbol{u}} \geq 0$ for $R \in [a, R_0]$.
Step 2. Using the fact that $\hat{\boldsymbol{u}}(\boldsymbol{x})$ is of the form (6.1.3), that

$$\psi(R) = \frac{k}{a^2} \log \left(\frac{R}{a} \right) \qquad \text{for } R \in [a, R_0]$$

and that

$$\hat{u}^i_{,\alpha\alpha} = \left(\frac{\partial \hat{u}^i}{\partial R} \frac{\partial R}{\partial x_\alpha} + \frac{\partial \hat{u}^i}{\partial \theta} \frac{\partial \theta}{\partial x_\alpha} \right)_{,\alpha},$$

it can be shown that for $R \in [a, R_0]$

$$\Delta \hat{\boldsymbol{u}} = - \frac{a}{R^3} \left(\frac{k^2}{a^4} + 1 \right) \begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix}. \tag{6.1.10}$$

Hence for $\boldsymbol{\phi}(\boldsymbol{x})$ of the form (6.1.4) we have

$$-\boldsymbol{\phi}.\Delta\hat{\boldsymbol{u}} = \begin{cases} -\Pi\left\{-\frac{a}{R^3}\left(\frac{k^2}{a^4}+1\right)\right\} & \text{for } R \in [a, R_0] \\ 0 & \text{for } R \in (R_0, b]. \end{cases}$$

By Proposition 6.1.2, $\Pi \geq 0$ holds for $R \in [a, R_0]$ since $\boldsymbol{\phi}(\boldsymbol{x})$ is an admissible variation. Thus $-\boldsymbol{\phi}.\Delta\hat{\boldsymbol{u}} \geq 0$ on $B_b \setminus B_a$, and so

$$\int_A \frac{1}{2}|\nabla(\hat{\boldsymbol{u}} + \epsilon\boldsymbol{\phi})|^2 \,\mathrm{dx} \geq \int_A \frac{1}{2}|\nabla\hat{\boldsymbol{u}}|^2 \,\mathrm{dx} + \epsilon^2 \int_A \frac{1}{2}|\nabla\boldsymbol{\phi}|^2 \,\mathrm{dx}.$$

Hence $\hat{\boldsymbol{u}}$ of the form (6.1.3) where $\rho(R)$ is given by (6.1.1) and $\psi(R)$ is given by (6.1.2) minimises E in this more general class of admissible variations. □

6.2 The incompressible case: positivity of the second variation.

We now return to the case in Example 5.0.3 of an incompressible nonlinear elastic body occupying the annular region A. We consider the energy functional

$$E(\boldsymbol{u}) := \int_A \frac{1}{2}|\nabla\boldsymbol{u}|^2 \,\mathrm{dx} \tag{6.2.1}$$

defined for incompressible deformations $\boldsymbol{u} : \bar{A} \to \bar{A}$, that is those deformations satisfying the internal constraint

$$\det(\nabla\boldsymbol{u}) = 1 \text{ a.e. in } A. \tag{6.2.2}$$

Recall that if $\boldsymbol{u}$ is rotationally symmetric and of the form (4.0.5) then the constraint (6.2.2) implies that $\rho(R) \equiv (R^2+c)^{\frac{1}{2}}$ and the boundary conditions ensure that $\rho(R) \equiv R$. In Chapter 5 we showed the existence of a unique rotationally symmetric minimiser in $\mathcal{A}_N^{\mathrm{inc}}$ for each N and that each minimiser satisfied the equilibrium equations with a corresponding pressure term $P(R)$. In this section we show that each of the rotationally symmetric minimisers is a weak local minimiser in a class of general deformations. In order to show this we require the second variation of E around a rotationally symmetric deformation.

6.2.1 Derivation of the second variation around an equilibrium solution.

For $p \geq 2$, let $\mathcal{A}^{\text{inc}}$ be of the form

$$\mathcal{A}^{\text{inc}} := \{\boldsymbol{u} \in W^{1,p}(A) : \det(\nabla \boldsymbol{u}) = 1 \text{ a.e. in } A, \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x} \text{ on } \partial A\}. \tag{6.2.3}$$

Let $\bar{\boldsymbol{u}} \in \mathcal{A}^{\text{inc}}$ be a solution of the equilibrium equations

$$\frac{\partial}{\partial x^\alpha}\left[\frac{\partial W}{\partial F^i_\alpha}(\nabla \boldsymbol{u}(\boldsymbol{x})) - \bar{P}(\boldsymbol{x})(\text{adj}(\nabla \boldsymbol{u}(\boldsymbol{x})))^\alpha_i\right] = 0, \ \boldsymbol{x} \in A, \ i = 1, 2, \tag{6.2.4}$$

and consider a one-parameter family of incompressible deformations $\boldsymbol{u}_\varepsilon$, $\varepsilon \in (-\zeta, \zeta)$, satisfying $\boldsymbol{u}_0 \equiv \bar{\boldsymbol{u}}$ and $\boldsymbol{u}_\varepsilon|_{\partial A} \equiv \bar{\boldsymbol{u}}|_{\partial A}$, so that $\bar{\boldsymbol{u}}$ is embedded in a one-parameter family of incompressible deformations satisfying the same boundary conditions. We note that if (6.2.2) holds then

$$\text{adj}(\nabla \boldsymbol{u}) = (\nabla \boldsymbol{u})^{-1}. \tag{6.2.5}$$

We first give two results on the properties of the first and second derivatives of $\boldsymbol{u}_\varepsilon$ with respect to ε.

Lemma 6.2.1 *Let $\bar{\boldsymbol{u}}$ be embedded in a one-parameter family of incompressible deformations and let*

$$\boldsymbol{\varphi}(\boldsymbol{x}) = \left.\frac{\partial \boldsymbol{u}_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0}.$$

Then $\boldsymbol{\varphi}$ is in $\mathcal{T}(\bar{\boldsymbol{u}})$ where

$$\mathcal{T}(\bar{\boldsymbol{u}}) = \{\boldsymbol{f} \in C^1(A; \mathbb{R}^2) : \nabla \boldsymbol{f}.((\nabla \bar{\boldsymbol{u}})^{-1})^T = 0 \text{ a.e. in } A, \boldsymbol{f} = \boldsymbol{0} \text{ on } \partial A\}.$$

Proof: Differentiating $\det(\nabla \boldsymbol{u}_\varepsilon) = 1$ with respect to ε gives us that

$$(\text{adj}(\nabla \boldsymbol{u}_\varepsilon))^T.\nabla\left.\left(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial \varepsilon}\right)\right|_{\varepsilon=0} = 0$$

and by (6.2.5) the result follows. □

Proposition 6.2.2 *Let $\bar{\boldsymbol{u}}$ be embedded in a one-parameter family of incompressible deformations and let*

$$\boldsymbol{\tau}(\boldsymbol{x}) = \left.\frac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial \varepsilon^2}\right|_{\varepsilon=0}.$$

Then $\boldsymbol{\tau}$ is in $\mathcal{S}(\bar{\boldsymbol{u}};\boldsymbol{\varphi})$ where

$$\mathcal{S}(\bar{\boldsymbol{u}};\boldsymbol{\varphi}) = \{\boldsymbol{g} \in C^1(A;\mathbb{R}^2) : \\ \nabla \boldsymbol{g}.((\nabla\bar{\boldsymbol{u}})^{-1})^T - \nabla\boldsymbol{\varphi}.((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi}(\nabla\bar{\boldsymbol{u}})^{-1})^T = 0 \textit{ in } A, \boldsymbol{g} = \boldsymbol{0} \textit{ on } \partial A\}.$$

Proof: Differentiating $\det(\nabla \boldsymbol{u}_\varepsilon) = 1$ twice with respect to ε gives us that

$$\frac{\partial}{\partial\varepsilon}(\operatorname{adj}(\nabla\boldsymbol{u}_\varepsilon))^T.\nabla\left(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0} + (\operatorname{adj}(\nabla\boldsymbol{u}_\varepsilon))^T.\nabla\left(\frac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial\varepsilon^2}\right)\bigg|_{\varepsilon=0} = 0. \tag{6.2.6}$$

Using (6.2.5) we can rewrite (6.2.6) as

$$\frac{\partial}{\partial\varepsilon}(((\nabla\boldsymbol{u}_\varepsilon)^{-1})^T).\nabla\left(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial\varepsilon}\right)\bigg|_{\varepsilon=0} + ((\nabla\boldsymbol{u}_\varepsilon)^{-1})^T.\nabla\left(\frac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial\varepsilon^2}\right)\bigg|_{\varepsilon=0} = 0. \tag{6.2.7}$$

Now $\nabla\boldsymbol{u}_\varepsilon(\nabla\boldsymbol{u}_\varepsilon)^{-1} = \mathrm{Id}$. Thus $\nabla\dot{\boldsymbol{u}}_\varepsilon(\nabla\boldsymbol{u}_\varepsilon)^{-1} + \nabla\boldsymbol{u}_\varepsilon(\nabla\dot{\boldsymbol{u}}_\varepsilon)^{-1} = 0$, where "$\dot{}$" denotes differentiation with respect to ε. Thus, $-(\nabla\boldsymbol{u}_\varepsilon)^{-1}\nabla\dot{\boldsymbol{u}}_\varepsilon(\nabla\boldsymbol{u}_\varepsilon)^{-1} = (\nabla\dot{\boldsymbol{u}}_\varepsilon)^{-1}$ so that in particular $-(\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi}(\nabla\bar{\boldsymbol{u}})^{-1} = ((\nabla\dot{\bar{\boldsymbol{u}}})^{-1})$ and so we can rewrite (6.2.7) as

$$-((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi}(\nabla\bar{\boldsymbol{u}})^{-1})^T.\nabla\boldsymbol{\varphi} + \left\{((\nabla\boldsymbol{u}_\varepsilon)^{-1})^T.\nabla\left(\frac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial\varepsilon^2}\right)\right\}\bigg|_{\varepsilon=0} = 0 \tag{6.2.8}$$

and the result is obtained on putting $\boldsymbol{\tau}(\boldsymbol{x}) = \dfrac{\partial^2\boldsymbol{u}_\varepsilon}{\partial\varepsilon^2}\bigg|_{\varepsilon=0}$. □

The results that $\boldsymbol{\varphi} \in \mathcal{T}(\bar{\boldsymbol{u}})$ and $\boldsymbol{\tau} \in \mathcal{S}(\bar{\boldsymbol{u}};\boldsymbol{\varphi})$ can be found in [30]. We now recall the forms of the first and second variations of E around $\bar{\boldsymbol{u}}$. The first variation of E around $\bar{\boldsymbol{u}}$ is

$$\frac{d}{d\varepsilon}E(\boldsymbol{u}_\varepsilon)\bigg|_{\varepsilon=0} = \delta E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi}) = \int_A \frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}).\nabla\boldsymbol{\varphi}\,\mathrm{dx}, \tag{6.2.9}$$

or in index notation

$$\delta E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi}) = \int_A \frac{\partial W}{\partial F^i_\alpha}(\nabla\bar{\boldsymbol{u}})\varphi^i_{,\alpha}\,\mathrm{dx}, \tag{6.2.10}$$

and the second variation of E around $\bar{\boldsymbol{u}}$ is

$$\frac{d^2}{d\varepsilon^2}E(\boldsymbol{u}_\varepsilon)\bigg|_{\varepsilon=0} = \delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_A \left(\frac{\partial^2 W}{\partial F^2}(\nabla\bar{\boldsymbol{u}})\nabla\boldsymbol{\varphi}\right).\nabla\boldsymbol{\varphi} + \frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}).\nabla\boldsymbol{\tau}\,\mathrm{dx}, \tag{6.2.11}$$

or in index notation

$$\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_A \frac{\partial^2 W}{\partial F^i_\alpha \partial F^j_\beta}(\nabla\bar{\boldsymbol{u}})\varphi^i_{,\alpha}\varphi^j_{,\beta} + \frac{\partial W}{\partial F^i_\alpha}(\nabla\bar{\boldsymbol{u}})\tau^i_{,\alpha}\ \mathrm{dx}. \tag{6.2.12}$$

We next show that the second variation of $E(\boldsymbol{u})$ around $\bar{\boldsymbol{u}}$ can be written in a form that is independent of $\boldsymbol{\tau}$. In order to do this we require the following results.

Lemma 6.2.3 *For $\nabla\bar{\boldsymbol{u}}$, $\nabla\boldsymbol{\varphi} \in M^{2\times 2}$ such that*

$$((\nabla\bar{\boldsymbol{u}})^{-1})^T.\nabla\boldsymbol{\varphi} = 0 \tag{6.2.13}$$

and $\det(\nabla\bar{\boldsymbol{u}}) = 1$, *we can write*

$$\mathrm{tr}((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi})^2 = -2\det(\nabla\boldsymbol{\varphi}).$$

Proof: For $\nabla\bar{\boldsymbol{u}}$, $\nabla\boldsymbol{\varphi} \in M^{2\times 2}$,

$$\begin{aligned}
&\mathrm{tr}((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi})^2 \\
&\quad = (u^2_{,2}\varphi^1_{,1} - u^1_{,2}\varphi^2_{,1})^2 + 2(u^2_{,2}\varphi^1_{,2} - u^1_{,2}\varphi^2_{,2})(u^1_{,1}\varphi^2_{,1} - u^2_{,1}\varphi^1_{,1}) + (u^1_{,1}\varphi^2_{,2} - u^2_{,1}\varphi^1_{,2})^2 \\
&\quad = (u^2_{,2}\varphi^1_{,1} - u^1_{,2}\varphi^2_{,1} - u^2_{,1}\varphi^1_{,2} + u^1_{,1}\varphi^2_{,2})^2 - 2(u^2_{,2}u^1_{,1} - u^1_{,2}u^2_{,1})(\varphi^2_{,2}\varphi^1_{,1} - \varphi^1_{,2}\varphi^2_{,1}).
\end{aligned}$$

Now, we have that $u^2_{,2}\varphi^1_{,1} - u^1_{,2}\varphi^2_{,1} - u^2_{,1}\varphi^1_{,2} + u^1_{,1}\varphi^2_{,2} = 0$ and $u^2_{,2}u^1_{,1} - u^1_{,2}u^2_{,1} = 1$. Thus

$$\begin{aligned}
(u^2_{,2}\varphi^1_{,1} - u^1_{,2}\varphi^2_{,1})^2 + 2(u^2_{,2}\varphi^1_{,2} - u^1_{,2}\varphi^2_{,2})(u^1_{,1}\varphi^2_{,1} - u^2_{,1}\varphi^1_{,1}) + (u^1_{,1}\varphi^2_{,2} - u^2_{,1}\varphi^1_{,2})^2 \\
= -2(\varphi^2_{,2}\varphi^1_{,1} - \varphi^1_{,2}\varphi^2_{,1})
\end{aligned}$$

and so $\mathrm{tr}((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi})^2 = -2\det(\nabla\boldsymbol{\varphi})$ as required. □

Proposition 6.2.4 *Suppose that $\bar{\boldsymbol{u}} \in \bar{\mathcal{A}}$ is a C^2 solution of*

$$\int_A \frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}).\nabla\boldsymbol{\varphi}\ \mathrm{dx} = 0 \tag{6.2.14}$$

for $\boldsymbol{\varphi}$ in $\mathcal{T}(\bar{\boldsymbol{u}})$. Then there exists a function $\bar{P} : A \to \mathbb{R}$ which is C^1 and is such that

$$\mathrm{Div}\left[\frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}})\right] = ((\nabla\bar{\boldsymbol{u}})^{-1})^T\nabla\bar{P}. \tag{6.2.15}$$

(For a proof see [30].)

The next result shows that the second variation can be rewritten in a form independent of $\boldsymbol{\tau}$.

Proposition 6.2.5 *Let $\bar{\boldsymbol{u}}$ be a solution of the equilibrium equations (6.2.15) and let $\boldsymbol{\varphi}$ be in $\mathcal{T}(\bar{\boldsymbol{u}})$. Then the second variation can be expressed as*

$$\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_A \left(\frac{\partial^2 W}{\partial F^2}(\nabla\bar{\boldsymbol{u}})\nabla\boldsymbol{\varphi}\right).\nabla\boldsymbol{\varphi} - 2\bar{P}(\boldsymbol{x})\det(\nabla\boldsymbol{\varphi})\ \mathrm{dx}. \qquad (6.2.16)$$

Proof: The proof follows the argument of [30] pp 281-82. From (6.2.15) we have $\operatorname{Div}\left(\frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}) - \bar{P}((\nabla\bar{\boldsymbol{u}})^{-1})^T\right) = \mathbf{0}$. Now $\boldsymbol{\varphi} \in \mathcal{T}(\bar{\boldsymbol{u}})$ and $\boldsymbol{\tau} \in \mathcal{S}(\bar{\boldsymbol{u}};\boldsymbol{\varphi})$, and so we have, on integrating by parts, that

$$\begin{aligned}
&\int_A \boldsymbol{\tau}.\left\{\operatorname{Div}\left(\frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}) - \bar{P}(\boldsymbol{x})((\nabla\bar{\boldsymbol{u}})^{-1})^T\right)\right\}\ \mathrm{dx}\\
&\quad = \int_A \frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}).\nabla\boldsymbol{\tau} - \bar{P}(\boldsymbol{x})\nabla\boldsymbol{\varphi}.((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi}(\nabla\bar{\boldsymbol{u}})^{-1})^T\ \mathrm{dx} = 0.
\end{aligned}$$

Hence for $\bar{\boldsymbol{u}}$ satisfying the equilibrium equations (6.2.15) and $\boldsymbol{\varphi} \in \mathcal{T}(\bar{\boldsymbol{u}})$ we have that

$$\int_A \frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}}).\nabla\boldsymbol{\tau}\ \mathrm{dx} = \int_A \bar{P}(\boldsymbol{x})\nabla\boldsymbol{\varphi}.((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi}(\nabla\bar{\boldsymbol{u}})^{-1})^T\ \mathrm{dx}. \qquad (6.2.17)$$

Hence we can rewrite the second variation (6.2.11) using (6.2.17) to obtain

$$\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_A \left(\frac{\partial^2 W}{\partial F^2}(\nabla\bar{\boldsymbol{u}})\nabla\boldsymbol{\varphi}\right).\nabla\boldsymbol{\varphi} + \bar{P}(\boldsymbol{x})\ \operatorname{tr}((\nabla\bar{\boldsymbol{u}})^{-1}\nabla\boldsymbol{\varphi})^2\ \mathrm{dx}. \qquad (6.2.18)$$

By Lemma 6.2.3 we can rewrite (6.2.18) as

$$\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_A \left(\frac{\partial^2 W}{\partial F^2}(\nabla\bar{\boldsymbol{u}})\nabla\boldsymbol{\varphi}\right).\nabla\boldsymbol{\varphi} - 2\bar{P}(\boldsymbol{x})\ \det(\nabla\boldsymbol{\varphi})\ \mathrm{dx} \qquad (6.2.19)$$

as required. □

Example 6.2.6 *For E of the form (6.2.1) the second variation of E around $\bar{\boldsymbol{u}}$ is*

$$\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_A |\nabla\boldsymbol{\varphi}|^2 - 2\bar{P}(\boldsymbol{x})\det(\nabla\boldsymbol{\varphi})\ \mathrm{dx}. \qquad (6.2.20)$$

6.2.2 The second variation around a rotationally symmetric solution.

We now return to the example of Chapter 5 where $E(\boldsymbol{u})$ is given by (6.2.1) for deformations $\boldsymbol{u}$ in $\mathcal{A}^{\text{inc}}$ (and thus the second variation is given by (6.2.20)). Throughout the rest of this section we will consider $\bar{\boldsymbol{u}}$ to be rotationally symmetric and of the form

$$\bar{\boldsymbol{u}}(\boldsymbol{x}) = R\begin{pmatrix} \cos(\theta + \bar{\psi}(R)) \\ \sin(\theta + \bar{\psi}(R)) \end{pmatrix} \tag{6.2.21}$$

where

$$\bar{\psi}(R) = \frac{2N\pi a^2 b^2}{b^2 - a^2}\left(\frac{1}{a^2} - \frac{1}{R^2}\right) \tag{6.2.22}$$

with $N \in \mathbb{N}$ being the number of twists of the deformation. We write any variation $\boldsymbol{\varphi} \in \mathcal{T}(\bar{\boldsymbol{u}})$ in the form

$$\boldsymbol{\varphi}(\boldsymbol{x}) = \phi(R,\theta)\begin{pmatrix} \cos(\theta + \bar{\psi}(R)) \\ \sin(\theta + \bar{\psi}(R)) \end{pmatrix} + \sigma(R,\theta)\begin{pmatrix} -\sin(\theta + \bar{\psi}(R)) \\ \cos(\theta + \bar{\psi}(R)) \end{pmatrix} \tag{6.2.23}$$

where $\phi(R,\theta)$ and $\sigma(R,\theta)$ satisfy $\phi(a,\theta) = \phi(b,\theta) = 0$, $\sigma(a,\theta) = \sigma(b,\theta) = 0$ for all θ and $\phi(R,0) = \phi(R,2\pi)$, $\sigma(R,0) = \sigma(R,2\pi)$ for all $R \in [a,b]$. We first rewrite the condition for $\boldsymbol{\varphi}$ to be in $\mathcal{T}(\bar{\boldsymbol{u}})$ for the case where $\bar{\boldsymbol{u}}$ is of the form (6.2.21) and $\boldsymbol{\varphi}$ is of the form (6.2.23).

Lemma 6.2.7 *Let $\bar{\boldsymbol{u}}$ be of the form (6.2.21). If $\boldsymbol{\varphi}$ of the form (6.2.23) is in $\mathcal{T}(\bar{\boldsymbol{u}})$ then*

$$\phi + \frac{\partial \sigma}{\partial \theta} + R\frac{\partial \phi}{\partial R} - R\frac{d\bar{\psi}}{dR}\frac{\partial \phi}{\partial \theta} = 0. \tag{6.2.24}$$

We also note that

$$\bar{P}(\boldsymbol{x}) = P(R) = \frac{4N^2\pi^2 a^4 b^4}{(b^2 - a^2)^2 R^4}. \tag{6.2.25}$$

We now present the second variation of E around $\bar{\boldsymbol{u}}$ of the form (6.2.21) for $\boldsymbol{\phi}$ of the form (6.2.23).

Proposition 6.2.8 *Let $\bar{\boldsymbol{u}}$ be of the form (6.2.21) and let $\boldsymbol{\varphi} \in \mathcal{T}(\bar{\boldsymbol{u}})$ be of the* *form (6.2.23). Then the second variation of E around $\bar{\mathbf{u}}$ is*

$$\delta^2 E(\bar{\mathbf{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{\left(\frac{\partial\phi}{\partial R}\right)^2 + (\phi\bar{\psi}')^2 + \left(\frac{\partial\sigma}{\partial R}\right)^2 + (\sigma\bar{\psi}')^2 \right.$$
$$- 2\frac{\partial\phi}{\partial R}\sigma\bar{\psi}' + 2\frac{\partial\sigma}{\partial R}\phi\bar{\psi}' + \left(\frac{1}{R}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\phi}{R}\right)^2 + \left(\frac{1}{R}\frac{\partial\sigma}{\partial\theta}\right)^2 + \left(\frac{\sigma}{R}\right)^2$$
$$\left. - \frac{2}{R^2}\sigma\frac{\partial\phi}{\partial\theta} + \frac{2}{R^2}\phi\frac{\partial\sigma}{\partial\theta}\right\} - \frac{32N^2\pi^2a^4b^4}{(b^2-a^2)^2R^5}\left\{\frac{\phi^2}{2} + \frac{\sigma^2}{2} + \phi\frac{\partial\sigma}{\partial\theta}\right\} d\theta\, dR. \tag{6.2.26}$$

Proof: Substitution of (6.2.21) and (6.2.23) into (6.2.20) gives

$$\delta^2 E(\bar{\mathbf{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi})$$
$$= \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{\left(\frac{\partial\phi}{\partial R}\right)^2 + (\phi\bar{\psi}')^2 + \left(\frac{\partial\sigma}{\partial R}\right)^2 + (\sigma\bar{\psi}')^2 - 2\frac{\partial\phi}{\partial R}\sigma\bar{\psi}' + 2\frac{\partial\sigma}{\partial R}\phi\bar{\psi}' \right.$$
$$\left. + \left(\frac{1}{R}\frac{\partial\phi}{\partial\theta}\right)^2 + \left(\frac{\phi}{R}\right)^2 + \left(\frac{1}{R}\frac{\partial\sigma}{\partial\theta}\right)^2 + \left(\frac{\sigma}{R}\right)^2 - \frac{2}{R^2}\sigma\frac{\partial\phi}{\partial\theta} + \frac{2}{R^2}\phi\frac{\partial\sigma}{\partial\theta}\right\}$$
$$- 2\bar{P}(\boldsymbol{x}(R,\theta))\left\{\phi\frac{\partial\phi}{\partial R} - \bar{\psi}'\sigma\frac{\partial\sigma}{\partial\theta} + \frac{\partial\phi}{\partial R}\frac{\partial\sigma}{\partial\theta} - \frac{\partial\sigma}{\partial R}\frac{\partial\phi}{\partial\theta} - \bar{\psi}'\phi\frac{\partial\phi}{\partial\theta} + \sigma\frac{\partial\sigma}{\partial R}\right\} d\theta\, dR. \tag{6.2.27}$$

To obtain (6.2.26) note that as $\bar{P}(\boldsymbol{x})$ given by (6.2.25) is independent of θ by integrating by parts, noting the boundary conditions on ϕ and σ, we have

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} -2\bar{P}(\boldsymbol{x})\left\{\phi\frac{\partial\phi}{\partial R} - \bar{\psi}'\sigma\frac{\partial\sigma}{\partial\theta} + \frac{\partial\phi}{\partial R}\frac{\partial\sigma}{\partial\theta} - \frac{\partial\sigma}{\partial R}\frac{\partial\phi}{\partial\theta} - \bar{\psi}'\phi\frac{\partial\phi}{\partial\theta} + \sigma\frac{\partial\sigma}{\partial R}\right\} d\theta\, dR$$
$$= \int_{R=a}^{b}\int_{\theta=0}^{2\pi} 2\frac{dP(R)}{dR}\left\{\frac{\phi^2}{2} + \frac{\sigma^2}{2} + \phi\frac{\partial\sigma}{\partial\theta}\right\} d\theta\, dR$$

and as $\dfrac{dP}{dR}(R) = -\dfrac{16N^2\pi^2a^4b^4}{(b^2-a^2)^2R^5}$ the second variation can be rewritten as (6.2.26) as required. □

6.2.3 Positivity of the second variation around a rotationally symmetric solution.

We will now show that the second variation $\delta^2 E(\bar{\mathbf{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi})$ is positive definite, that is $\delta^2 E(\bar{\mathbf{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) > 0$ for a general class of variations $\boldsymbol{\varphi} \in \mathcal{T}(\bar{\mathbf{u}})$, $\boldsymbol{\varphi} \not\equiv 0$. We will assume the following hypotheses throughout this subsection:

(I1) $\phi(R,\theta)$ and $\sigma(R,\theta)$ satisfy (6.2.24);

(I2) $\phi(R,\theta)$ and $\sigma(R,\theta)$ are 2π-periodic in θ.

As a consequence of (I1) integration of (6.2.24) with respect to θ gives

$$\sigma(R,\theta) = -\int_{t=0}^{\theta} \left(\phi(R,t) + R\frac{\partial \phi}{\partial R}(R,t)\right) dt + R\frac{d\psi}{dR}(R)\phi(R,\theta) + C(R) \quad (6.2.28)$$

and the boundary conditions $\phi(a,\theta) = \phi(b,\theta) = 0$, $\sigma(a,\theta) = \sigma(b,\theta) = 0$ result in $C(a) = C(b) = 0$.
As a consequence of (I2) and the boundary conditions $\phi(a,\theta) = 0$, $\phi(b,\theta) = 0$ we must have

$$\int_{\theta=0}^{2\pi} \phi(R,\theta)\ d\theta = 0. \quad (6.2.29)$$

We now define $L(R,\theta)$ to be

$$L(R,\theta) := \int_{t=0}^{\theta} \phi(R,t)\ dt - \frac{1}{2\pi}\int_{s=0}^{2\pi}\int_{t=0}^{s} \phi(R,t)\ dt\ ds. \quad (6.2.30)$$

Then $\dfrac{\partial L}{\partial \theta}(R,\theta) = \phi(R,\theta)$ and we assume that

$$\int_{\theta=0}^{2\pi} L(R,\theta)\ d\theta = 0. \quad (6.2.31)$$

We can write $L(R,\theta)$ as

$$L(R,\theta) = \int_{t=0}^{\theta} \phi(R,t)\ dt + g(R) \quad (6.2.32)$$

and so we have that

$$\sigma(R,\theta) = -\left\{L(R,\theta) + R\frac{\partial L}{\partial R}(R,\theta)\right\} + R\frac{d\psi}{dR}(R)\phi(R,\theta) + D(R) \quad (6.2.33)$$

where $D(R) = C(R) - g(R) - Rg'(R)$. Also by (6.2.33) we must have that $L(R,\theta)$ and $\dfrac{\partial L}{\partial R}(R,\theta)$ are also 2π-periodic in θ.
We now present the main result in this section.

Proposition 6.2.9 *Let $\bar{\mathbf{u}}(\boldsymbol{x})$ be a rotationally symmetric equilibrium solution of the form (6.2.21) (with $\bar{\psi}(R)$ of the form (6.2.22)). Then $\bar{\mathbf{u}}(\boldsymbol{x})$ is a weak local minimiser among admissible deformations satisfying the constraint* $\det(\nabla \boldsymbol{u}) =$

1, *where the variations* $\boldsymbol{\varphi}$ *given by (6.2.23) are such that* $\phi(R,\theta)$ *and* $\sigma(R,\theta)$ *satisfy conditions* (I1) *and* (I2) *and* $L(R,\theta)$ *is defined by (6.2.30), and the second variation is given by*

$$\begin{aligned}
&\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) \\
&= \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left(\frac{\partial\phi}{\partial R}\right)^2 + \frac{1}{R}\left\{\left(\frac{\partial\phi}{\partial\theta}-\sigma\right)^2 + \left(\phi+\frac{\partial\sigma}{\partial\theta}\right)^2\right\} \\
&+ \frac{2k^2}{R^5}\phi^2 + R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\bar{\psi}}{dR}\right] - R\frac{\partial^2 L}{\partial R^2}\right)^2 + R\left(\frac{dD}{dR}\right)^2 dR\, d\theta. \qquad (6.2.34)
\end{aligned}$$

Proof: In order to show the result we are required to show that the second variation $\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi})$ is positive definite. In order to achieve this we will show that the second variation can be written as a sum of squares. The method of proof is by a lengthy but straightforward calculation. Now the second variation of E around $\bar{\boldsymbol{u}}$ is

$$\begin{aligned}
\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_{R=a}^{b}\int_{\theta=0}^{2\pi} & R\left(\frac{\partial\phi}{\partial R}\right)^2 + R(\phi\bar{\psi}')^2 + R\left(\frac{\partial\sigma}{\partial R}\right)^2 + R(\sigma\bar{\psi}')^2 \\
& -2R\frac{\partial\phi}{\partial R}\sigma\bar{\psi}' + 2R\frac{\partial\sigma}{\partial R}\phi\bar{\psi}' + \frac{1}{R}\left\{\left(\frac{\partial\phi}{\partial\theta}-\sigma\right)^2 + \left(\phi+\frac{\partial\sigma}{\partial\theta}\right)^2\right\} \\
& + P'(R)\left\{\phi^2+\sigma^2+2\phi\frac{\partial\sigma}{\partial\theta}\right\} d\theta\, dR.
\end{aligned} \qquad (6.2.35)$$

The expression

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left(\frac{\partial\phi}{\partial R}\right)^2 + \frac{1}{R}\left\{\left(\frac{\partial\phi}{\partial\theta}-\sigma\right)^2 + \left(\phi+\frac{\partial\sigma}{\partial\theta}\right)^2\right\} d\theta\, dR$$

appears in both (6.2.34) and (6.2.35). Therefore, we are required to show that

$$\begin{aligned}
&\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R(\phi\bar{\psi}')^2 + R\left(\frac{\partial\sigma}{\partial R}\right)^2 + R(\sigma\bar{\psi}')^2 - 2R\frac{\partial\phi}{\partial R}\sigma\bar{\psi}' + 2R\frac{\partial\sigma}{\partial R}\phi\bar{\psi}' \\
&\qquad + P'(R)\left\{\phi^2+\sigma^2+2\phi\frac{\partial\sigma}{\partial\theta}\right\} d\theta\, dR \\
&= \int_{R=a}^{b}\int_{\theta=0}^{2\pi} \frac{2k^2}{R^5}\phi^2 + R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\bar{\psi}}{dR}\right] - R\frac{\partial^2 L}{\partial R^2}\right)^2 + R\left(\frac{dD}{dR}\right)^2 d\theta\, dR
\end{aligned} \qquad (6.2.36)$$

holds. We define

$$F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) := \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R(\phi\bar{\psi}')^2 + R\left(\frac{\partial\sigma}{\partial R}\right)^2 + R(\sigma\bar{\psi}')^2 - 2R\frac{\partial\phi}{\partial R}\sigma\bar{\psi}' + 2R\frac{\partial\sigma}{\partial R}\phi\bar{\psi}' + P'(R)\left\{\phi^2+\sigma^2+2\phi\frac{\partial\sigma}{\partial\theta}\right\}\,d\theta\,dR. \tag{6.2.37}$$

From Example 5.0.3, $P'(R) = -R(\bar{\psi}'(R))^2$. Also from (6.2.24) we have

$$\frac{\partial\sigma}{\partial\theta} = -\phi - R\frac{\partial\phi}{\partial R} + R\frac{d\bar{\psi}}{dR}\frac{\partial\phi}{\partial\theta} \tag{6.2.38}$$

and thus

$$F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left(\frac{\partial\sigma}{\partial R}\right)^2 - 2R\frac{\partial\phi}{\partial R}\sigma\bar{\psi}' + 2R\frac{\partial\sigma}{\partial R}\phi\bar{\psi}' - 2R(\bar{\psi}')^2\phi\left\{2R\bar{\psi}'\frac{\partial\phi}{\partial\theta} - \phi - R\frac{\partial\phi}{\partial R}\right\}\,d\theta\,dR. \tag{6.2.39}$$

Now note that

$$\int_{\theta=0}^{2\pi} 2R^2(\bar{\psi}')^3\phi\frac{\partial\phi}{\partial\theta}\,d\theta = 0 \tag{6.2.40}$$

and that

$$\int_{R=a}^{b} -2R\bar{\psi}'\left\{\frac{\partial\phi}{\partial R}\sigma - \frac{\partial\sigma}{\partial R}\phi\right\}\,dR = \int_{R=a}^{b} -4R\bar{\psi}'\frac{\partial\phi}{\partial R}\sigma - 2(\bar{\psi}' + R\bar{\psi}'')\phi\sigma\,dR. \tag{6.2.41}$$

Substitution of (6.2.33) for σ, (6.2.40) and (6.2.41) into (6.2.39) gives us that

$$\begin{aligned} &F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) \\ &= \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{\frac{\partial}{\partial R}\left[D(R) + R\bar{\psi}'\phi - \left(L + R\frac{\partial L}{\partial R}\right)\right]\right\}^2 \\ &\quad - \left\{4R\bar{\psi}'\frac{\partial\phi}{\partial R} + 2\phi(\bar{\psi}' + R\bar{\psi}'')\right\}\left\{D(R) + R\bar{\psi}'\phi - \left(L + R\frac{\partial L}{\partial R}\right)\right\} \\ &\quad + 2R(\bar{\psi}')^2\phi^2 + 2R^2(\bar{\psi}')^2\phi\frac{\partial\phi}{\partial R}\,d\theta\,dR. \end{aligned} \tag{6.2.42}$$

Now

$$\int_{R=a}^{b} -R\bar{\psi}'\phi\left\{4R\bar{\psi}'\frac{\partial\phi}{\partial R}+2\phi(\bar{\psi}'+R\bar{\psi}'')\right\}+2R^2(\bar{\psi}')^2\phi\frac{\partial\phi}{\partial R}\,dR$$
$$=\int_{R=a}^{b} -2(R\bar{\psi}')^2\frac{\partial}{\partial R}\left[\frac{\phi^2}{2}\right]-2\phi^2\frac{d}{dR}\left[\frac{(R\bar{\psi}')^2}{2}\right]\,dR=0. \tag{6.2.43}$$

Also we can write

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{\frac{\partial}{\partial R}\left[D(R)+R\bar{\psi}'\phi-\left(L+R\frac{\partial L}{\partial R}\right)\right]\right\}^2\,d\theta\,dR$$
$$=\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{4\left(\frac{\partial L}{\partial R}\right)^2+\left[D'-R\frac{\partial^2 L}{\partial R^2}-\frac{\partial}{\partial R}[R\bar{\psi}'\phi]\right]^2\right.$$
$$\left.-4\left(\frac{\partial L}{\partial R}\right)\left[D'-R\frac{\partial^2 L}{\partial R^2}+(\bar{\psi}'+R\bar{\psi}'')\phi+R\bar{\psi}'\frac{\partial\phi}{\partial R}\right]\right\}\,d\theta\,dR. \tag{6.2.44}$$

Thus we have

$$F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi})=\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{4\left(\frac{\partial L}{\partial R}\right)^2+\left[D'-R\frac{\partial^2 L}{\partial R^2}-\frac{\partial}{\partial R}[R\bar{\psi}'\phi]\right]^2\right.$$
$$+2R(\bar{\psi}')^2\phi^2-\left\{4R\bar{\psi}'\frac{\partial\phi}{\partial R}+2\phi(\bar{\psi}'+R\bar{\psi}'')\right\}\left\{D(R)-\left(L+R\frac{\partial L}{\partial R}\right)\right\}$$
$$\left.-4\left(\frac{\partial L}{\partial R}\right)\left[D'-R\frac{\partial^2 L}{\partial R^2}+(\bar{\psi}'+R\bar{\psi}'')\phi+R\bar{\psi}'\frac{\partial\phi}{\partial R}\right]\right\}\,d\theta\,dR. \tag{6.2.45}$$

Consider the term

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} -\left\{4R\bar{\psi}'\frac{\partial\phi}{\partial R}+2\phi(\bar{\psi}'+R\bar{\psi}'')\right\}\left\{D(R)-\left(L+R\frac{\partial L}{\partial R}\right)\right\}\,d\theta\,dR \tag{6.2.46}$$

appearing in (6.2.45). As $D(R)$ and $\bar{\psi}'(R)$ are functions of R only, then using (6.2.29) we can write

$$\int_{\theta=0}^{2\pi} D\left\{4R\bar{\psi}'\frac{\partial\phi}{\partial R}+2(\bar{\psi}'+R\bar{\psi}'')\phi\right\}\,d\theta=0.$$

Also

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} -\left\{4R\bar{\psi}'\frac{\partial \phi}{\partial R}+2\phi(\bar{\psi}'+R\bar{\psi}'')\right\}\left(L+R\frac{\partial L}{\partial R}\right)$$
$$-4R\left(\frac{\partial L}{\partial R}\right)\left[D'-R\frac{\partial^2 L}{\partial R^2}+(\bar{\psi}'+R\bar{\psi}'')\phi+R\bar{\psi}'\frac{\partial \phi}{\partial R}\right]\,d\theta\,dR$$
$$=\int_{R=a}^{b}\int_{\theta=0}^{2\pi} -4RD'\frac{\partial L}{\partial R}+4R\bar{\psi}'L\frac{\partial}{\partial R}\left(\frac{\partial L}{\partial \theta}\right)+4R^2\bar{\psi}'\frac{\partial L}{\partial R}\frac{\partial}{\partial R}\left(\frac{\partial L}{\partial \theta}\right)$$
$$+2(\bar{\psi}'+R\bar{\psi}'')L\frac{\partial L}{\partial \theta}-4R^2\bar{\psi}'\frac{\partial L}{\partial R}\frac{\partial}{\partial R}\left(\frac{\partial L}{\partial \theta}\right)+2R(\bar{\psi}'+R\bar{\psi}'')\frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}$$
$$+4R^2\frac{\partial L}{\partial R}\frac{\partial^2 L}{\partial R^2}-4R(\bar{\psi}'+R\bar{\psi}'')\frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}\,d\theta\,dR$$
$$=\int_{R=a}^{b}\int_{\theta=0}^{2\pi} -4RD'\frac{\partial L}{\partial R}+4R\bar{\psi}'L\frac{\partial}{\partial R}\left(\frac{\partial L}{\partial \theta}\right)$$
$$+2(\bar{\psi}'+R\bar{\psi}'')L\frac{\partial L}{\partial \theta}-2R(\bar{\psi}'+R\bar{\psi}'')\frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}\,d\theta\,dR. \tag{6.2.47}$$

Therefore we can rewrite (6.2.45) as

$$F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi})$$
$$=\int_{R=a}^{b}\int_{\theta=0}^{2\pi} 2R(\bar{\psi}')^2\phi^2+R\left(D'-R\frac{\partial^2 L}{\partial R^2}-\frac{\partial}{\partial R}[R\bar{\psi}'\phi]\right)^2-4RD'\frac{\partial L}{\partial R}$$
$$+4R\bar{\psi}'L\frac{\partial}{\partial R}\left(\frac{\partial L}{\partial \theta}\right)+2(\bar{\psi}'+R\bar{\psi}'')L\frac{\partial L}{\partial \theta}-2R(\bar{\psi}'+R\bar{\psi}'')\frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}\,d\theta\,dR. \tag{6.2.48}$$

Recall that $\bar{\psi}'(R)=\dfrac{k}{R^3}$, where $k=\dfrac{4N\pi a^2b^2}{b^2-a^2}$. Thus $\bar{\psi}'(R)+R\bar{\psi}''(R)=-\dfrac{2k}{R^3}$. Now consider the term

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} 4R\bar{\psi}'L\frac{\partial}{\partial R}\left(\frac{\partial L}{\partial \theta}\right)+2(\bar{\psi}'+R\bar{\psi}'')L\frac{\partial L}{\partial \theta}-2R(\bar{\psi}'+R\bar{\psi}'')\frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}\,d\theta\,dR \tag{6.2.49}$$

in (6.2.48). We can rewrite (6.2.49) as

$$\int_{R=a}^{b}\int_{\theta=0}^{2\pi} 4\frac{k}{R^2}L\frac{\partial^2 L}{\partial R\partial \theta}-4\frac{k}{R^3}L\frac{\partial L}{\partial \theta}+4\frac{k}{R^2}\frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}\,d\theta\,dR$$
$$=\int_{R=a}^{b}\int_{\theta=0}^{2\pi} -4\frac{k}{R^3}L\frac{\partial L}{\partial \theta}\,d\theta\,dR$$

since

$$\int_{\theta=0}^{2\pi} \frac{\partial L}{\partial R}\frac{\partial L}{\partial \theta}\, d\theta = \int_{\theta=0}^{2\pi} -L\frac{\partial^2 L}{\partial R \partial \theta}\, d\theta$$

since from condition (I2) $L(R,\theta)$ is 2π-periodic. Hence (6.2.48) can be written as

$$\begin{aligned} F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \int_{R=a}^{b}\int_{\theta=0}^{2\pi} & \frac{2k^2}{R^5}\phi^2 - \frac{4k}{R^3}L\frac{\partial L}{\partial \theta} - 4R\frac{\partial L}{\partial R}\frac{dD}{dR} \\ & + R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\bar{\psi}}{dR}\right] - R\frac{\partial^2 L}{\partial R^2} + \frac{dD}{dR}\right)^2 \, dR\, d\theta. \end{aligned} \tag{6.2.50}$$

Now L satisfies (6.2.31). Hence since

$$\begin{aligned} &\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\bar{\psi}}{dR}\right] - R\frac{\partial^2 L}{\partial R^2} + \frac{dD}{dR}\right)^2 \, dR\, d\theta \\ &\quad = \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\bar{\psi}}{dR}\right] - R\frac{\partial^2 L}{\partial R^2}\right)^2 - 2R\frac{\partial^2 L}{\partial R^2}\frac{dD}{dR} + \left(\frac{dD}{dR}\right)^2 \, dR\, d\theta \end{aligned} \tag{6.2.51}$$

then we have that

$$\int_{\theta=0}^{2\pi} -4R\frac{\partial L}{\partial R}\frac{dD}{dR} - 2R\frac{\partial^2 L}{\partial R^2}\frac{dD}{dR} d\theta = 0$$

and, by periodicity of $L(R,\theta)$,

$$\int_{\theta=0}^{2\pi} \frac{4k}{R^3}L\frac{\partial L}{\partial \theta} d\theta = 0 \text{ and } \int_{\theta=0}^{2\pi} 2R\frac{\partial}{\partial R}\left[R\phi\frac{d\psi}{dR}\right]\frac{dD}{dR} d\theta = 0.$$

Thus

$$\begin{aligned} &F(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) \\ &\quad = \int_{R=a}^{b}\int_{\theta=0}^{2\pi} \frac{2k^2}{R^5}\phi^2 + R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\psi}{dR}\right] - R\frac{\partial^2 L}{\partial R^2}\right)^2 + R\left(\frac{dD}{dR}\right)^2 \, dR\, d\theta \end{aligned}$$

as required. Hence we have that

$$
\begin{aligned}
&\delta^2 E(\bar{\boldsymbol{u}})(\boldsymbol{\varphi},\boldsymbol{\varphi}) \\
&= \int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left(\frac{\partial\phi}{\partial R}\right)^2 + \frac{1}{R}\left\{\left(\frac{\partial\phi}{\partial\theta}-\sigma\right)^2 + \left(\phi+\frac{\partial\sigma}{\partial\theta}\right)^2\right\} \\
&+ \frac{2k^2}{R^5}\phi^2 + R\left(\frac{\partial}{\partial R}\left[R\phi\frac{d\psi}{dR}\right] - R\frac{\partial^2 L}{\partial R^2}\right)^2 + R\left(\frac{dD}{dR}\right)^2 \, dR\, d\theta
\end{aligned}
$$

and hence the second variation is positive definite as required. □

The next example shows that without the restriction (6.2.24) it does not necessarily follow that the second variation around an equilibrium solution is positive definite.

Example 6.2.10 *Put $N = 1$, $a = 1$ and $b = 4$ and put*

$$\phi(R,\theta) = (4-R)(R-1)\cos\left(\frac{32\pi}{15R^2}\right);$$

and

$$\sigma(R,\theta) = (4-R)(R-1)\sin\left(\frac{32\pi}{15R^2}\right).$$

Then

$$
\begin{aligned}
\phi + \frac{\partial\sigma}{\partial\theta} + R\frac{\partial\phi}{\partial R} - R\frac{d\psi}{dR}\frac{\partial\phi}{\partial\theta} = \frac{1}{15R^2}\Bigg[&(150R^3 - 60R^2 - 45R^4)\cos\left(\frac{32\pi}{15R^2}\right) \\
&+\pi(320R - 256 - 64R^2)\sin\left(\frac{32\pi}{15R^2}\right)\Bigg]
\end{aligned}
$$

which is not zero for all $R \in [a,b]$. Now an explicit expression for $\delta^2 E(\boldsymbol{u})(\boldsymbol{\varphi},\boldsymbol{\varphi})$ can be obtained and it can be shown that

$$\delta^2 E(\boldsymbol{u})(\boldsymbol{\varphi},\boldsymbol{\varphi}) = \pi\left(\frac{15}{2}+\frac{128}{3}\pi^2\right) + \pi\log(2)\left(64+\frac{16384}{225}\pi^2\right)$$

which (using Maple V) is approximately -25.19π.

6.3 Discussion of deformations of a slightly compressible annulus.

We now extend our analysis of §6.2 to consider deformations of a slightly compressible nonlinear elastic annulus. Mathematically, this corresponds to considering the energy functional $E_\delta(\boldsymbol{u})$ for an annulus composed of neo-Hookean material with $E_\delta(\boldsymbol{u})$ of the form

$$E_\delta(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}|^2 + \frac{1}{\delta}h(\det(\nabla \boldsymbol{u}))\ \mathrm{dx} \tag{6.3.1}$$

where δ is a parameter of compressibility, $\delta = 0$ corresponding to the incompressible case, and where h is C^4, convex and such that $h(d) \to \infty$ as $d \to 0, \infty$ with $h(1) = 0$. We look to see what can be said about the second variation of an energy functional around a solution of the equilibrium equations in the case of a slightly compressible annulus, given that the second variation of an energy functional around a solution of the equilibrium equations in the case of an incompressible annulus is positive definite. The question of convergence of a minimiser in the slightly compressible case to a minimiser in the incompressible case as the compressibility tends to zero has been discussed for other special cases. See, for example, [21], [45], [50] and [51]. The arguments of convergence of [21] and [45] follow from the existence results in [6]. It is also shown in [45, part B] that if $\bar{\boldsymbol{u}}_\delta$ is a solution of the equilibrium equations in the slightly compressible case then the hydrostatic pressure associated with the equilibrium equations in the incompressible problem can be written in terms of the limit of the Cauchy stress tensor corresponding to the slightly compressible case (see also [46]). This result is achieved in [45] by assuming that the solution of the equilibrium equations in the slightly compressible case $\bar{\boldsymbol{u}}_\delta$ is in $W^{2,p}(A)$, $p > 3$, and can be written as an expansion of the form

$$\bar{\boldsymbol{u}}_\delta = \bar{\boldsymbol{u}}_0 + \delta \boldsymbol{w}_1 + o(\delta),$$

where $\bar{\boldsymbol{u}}_0$, $\boldsymbol{w}_1 \in W^{2,p}(A)$, and it can be shown that $\bar{\boldsymbol{u}}_0$ is a solution of the equilibrium equations in the incompressible case (see [45, Theorem B II.2]). This assumption is partly justified in [45, part C] by showing the existence of an expansion in the case of a pure displacement problem, under suitable ellipticity assumptions on the incompressible Piola-Kirchhoff stress tensor at the identity mapping (which will ensure that the strong ellipticity condition is satisfied), and by use of the implicit function theorem.

We will adopt the approach in [45] by considering an energy functional of the form (6.3.1), letting $\bar{\boldsymbol{u}}_\delta$ be a solution of the equilibrium equations in the case of a slightly compressible elastic annulus and making the following assumptions on the deformations $\bar{\boldsymbol{u}}_\delta$ and the function $h(d)$ throughout the rest of this section:

(S1) $\bar{\boldsymbol{u}}_\delta \in C^2(\bar{A})$ can be written in terms of an expansion of the form

$$\bar{\boldsymbol{u}}_\delta = \bar{\boldsymbol{u}}_0 + \delta \boldsymbol{w}_1 + \delta^2 \boldsymbol{w}_2 + \dots \tag{6.3.2}$$

where $\bar{\boldsymbol{u}}_0 \in C^2(\bar{A})$ is the solution of the equilibrium equations in the incompressible case and $\boldsymbol{w}_i \in C^2(\bar{A})$ for each $i \in \mathbb{N}$;

(S2) $h(d) = 0$ if and only if $d = 1$, and $h'(d) = 0$ if and only if $d = 1$.

Throughout this section we will denote the second variation by $D^2 E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi}, \boldsymbol{\phi})$. Let $\boldsymbol{u}_\delta$ be a general deformation. If we put $\boldsymbol{u}_\delta(\boldsymbol{x}) = \bar{\boldsymbol{u}}_\delta(\boldsymbol{x}) + \varepsilon \boldsymbol{\phi}(\boldsymbol{x}) + o(\varepsilon)$ then the second variation $D^2 E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi}, \boldsymbol{\phi})$ is

$$\begin{aligned} &D^2 E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi}, \boldsymbol{\phi}) = \\ &\quad \int_A |\nabla \boldsymbol{\phi}|^2 + \frac{2}{\delta} h'(\det(\nabla \bar{\boldsymbol{u}}_\delta)) \det(\nabla \boldsymbol{\phi}) + \frac{1}{\delta} h''(\det(\nabla \bar{\boldsymbol{u}}_\delta))(\operatorname{adj}(\nabla \bar{\boldsymbol{u}}_\delta)^T . \nabla \boldsymbol{\phi})^2 \text{ dx}. \end{aligned} \tag{6.3.3}$$

It follows from (S1) and (S2) that

$$\begin{aligned} \int_A \frac{2}{\delta} h'(\det(\nabla \bar{\boldsymbol{u}}_\delta)) \det(\nabla \boldsymbol{\phi}) \text{ dx} &= \int_A 2 \left\{ h''(\det(\nabla \bar{\boldsymbol{u}}_0))(\operatorname{adj}(\nabla \bar{\boldsymbol{u}}_0)^T . \nabla \boldsymbol{w}_1) \right. \\ &\left. + \delta h'''(\det(\nabla \bar{\boldsymbol{u}}_0))(\operatorname{adj}(\nabla \bar{\boldsymbol{u}}_0)^T . \nabla \boldsymbol{w}_2 + \det(\nabla \boldsymbol{w}_1)) + O(\delta^2) \right\} \det(\nabla \boldsymbol{\phi}) \text{ dx}. \end{aligned} \tag{6.3.4}$$

It can also be shown, using [45, Theorem B II.2] and [46], that the hydrostatic pressure in the incompressible case $\bar{P}$ satisfies

$$\bar{P} = -h''(\det(\nabla \bar{\boldsymbol{u}}_0))((\operatorname{adj}(\nabla \bar{\boldsymbol{u}}_0))^T . \nabla \boldsymbol{w}_1). \tag{6.3.5}$$

Thus substituting (6.3.4) and (6.3.5) into (6.3.3) we have that

$$\begin{aligned} D^2 E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi}, \boldsymbol{\phi}) &= \int_A |\nabla \boldsymbol{\phi}|^2 + \frac{1}{\delta} h''(\det(\nabla \bar{\boldsymbol{u}}_\delta))(\operatorname{adj}(\nabla \bar{\boldsymbol{u}}_\delta)^T . \nabla \boldsymbol{\phi})^2 + 2 \left\{ -\bar{P}(\boldsymbol{x}) \right. \\ &\left. + \delta h'''(\det(\nabla \bar{\boldsymbol{u}}_0))(\operatorname{adj}(\nabla \bar{\boldsymbol{u}}_0)^T . \nabla \boldsymbol{w}_2 + \det(\nabla \boldsymbol{w}_1)) + O(\delta^2) \right\} \det(\nabla \boldsymbol{\phi}) \text{ dx}. \end{aligned} \tag{6.3.6}$$

In this section we will extend the result of positivity of the second variation to the slightly compressible case. We first show the following.

Proposition 6.3.1 *For each ϕ there exists a value of δ, $\bar{\delta}_{\phi}$ say, such that the second variation (6.3.3) is positive definite for $\delta < \bar{\delta}_{\phi}$.*

Proof: Choose a particular variation ϕ. If ϕ satisfies $\operatorname{adj}(\nabla\bar{\mathbf{u}}_0)^T.\nabla\phi = 0$ in A then clearly $\frac{1}{\delta}h''(\det(\nabla\bar{\mathbf{u}}_\delta))(\operatorname{adj}(\nabla\bar{\mathbf{u}}_\delta)^T.\nabla\phi)^2 \to 0$ as $\delta \to 0$ and the second variation becomes

$$D^2E_0(\bar{\mathbf{u}}_0)(\phi,\phi) = \int_A |\nabla\phi|^2 - 2\bar{P}(\boldsymbol{x})\det(\nabla\phi)\ \mathrm{dx}$$

which is positive definite. Otherwise $\frac{1}{\delta}h''(\det(\nabla\bar{\mathbf{u}}_\delta))(\operatorname{adj}(\nabla\bar{\mathbf{u}}_\delta)^T.\nabla\phi)^2 \to \infty$ as $\delta \to 0$ and thus $D^2E_0(\bar{\mathbf{u}}_0)(\phi,\phi) = \infty$. □

We now want to show uniformity of δ. We do this by showing that there is a value of δ, $\bar{\delta}$ say, such that the second variation $D^2E_\delta(\bar{\mathbf{u}}_\delta)(\phi,\phi)$ is $W^{1,2}$-positive definite for $\delta < \bar{\delta}$ and all ϕ, where $D^2E_\delta(\bar{\mathbf{u}}_\delta)(\phi,\phi)$ is given by (6.3.3). Uniformity of δ is shown by the following result:

Proposition 6.3.2 *Suppose that $\bar{\mathbf{u}}_\delta \in C^2(\bar{A})$ is a solution of the equilibrium equations for the energy functional*

$$E_\delta(\boldsymbol{u}_\delta) = \int_A \frac{1}{2}|\nabla\boldsymbol{u}_\delta|^2 + \frac{1}{\delta}h(\det(\nabla\boldsymbol{u}_\delta))\ \mathrm{dx}$$

where δ is a parameter of compressibility, $\delta = 0$ corresponding to the incompressible case, and where $\bar{\mathbf{u}}_\delta$ satisfies (S1) *and h satisfies* (S2). *Suppose that the second variation around $\bar{\mathbf{u}}_0$ in the class of incompressible maps is positive definite, that is for all $\phi \not\equiv \mathbf{0}$ satisfying* $(\operatorname{adj}(\nabla\bar{\mathbf{u}}_0))^T.\nabla\phi = 0$ *a.e.*

$$D^2E_0(\bar{\mathbf{u}}_0)(\phi,\phi) = \int_A |\nabla\phi|^2 - 2\bar{P}(\boldsymbol{x})\det(\nabla\phi)\ \mathrm{dx} > 0, \qquad (6.3.7)$$

where $\bar{P}(\boldsymbol{x})$ is the hydrostatic pressure corresponding to the solution $\bar{\mathbf{u}}_0$. Then there exists $\bar{\delta} > 0$ such that for all $\phi \in W_0^{1,2}(A)$ and $0 < \delta < \bar{\delta}$ the second variation of $E(\bar{\mathbf{u}}_\delta)$ around $\bar{\mathbf{u}}_\delta$ is $W^{1,2}$-positive definite, that is for each $\delta \in (0,\bar{\delta})$

there exists $C_\delta > 0$ such that for all $\boldsymbol{\phi} \not\equiv \mathbf{0}$

$$D^2E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi},\boldsymbol{\phi}) =$$
$$\int_A |\nabla\boldsymbol{\phi}|^2 + \frac{2}{\delta}h'(\det(\nabla\bar{\boldsymbol{u}}_\delta))\det(\nabla\boldsymbol{\phi}) + \frac{1}{\delta}h''(\det(\nabla\bar{\boldsymbol{u}}_\delta))(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_\delta)^T.\nabla\boldsymbol{\phi})^2 \mathrm{dx}$$
$$> C_\delta \parallel \boldsymbol{\phi} \parallel_{1,2}^2 . \tag{6.3.8}$$

Proof: Suppose (for a contradiction) that given any $\bar{\delta} > 0$ there exist $\delta \in (0, \bar{\delta})$ and $\boldsymbol{\phi} \in W_0^{1,2}(A)$ such that $D^2E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi},\boldsymbol{\phi}) \leq \delta \parallel \boldsymbol{\phi} \parallel_{1,2}^2$.
The approach we will take in this proof is as follows: we will consider a sequence (δ_n) such that $\delta_n \to 0$ as $n \to \infty$ and a sequence $(\boldsymbol{\phi}_n) \in W_0^{1,2}(A)$ such that $D^2E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi}_n,\boldsymbol{\phi}_n) \leq \delta_n \parallel \boldsymbol{\phi}_n \parallel_{1,2}^2$, and we will show the existence of a weakly convergent subsequence $(\boldsymbol{\phi}_n) \in W_0^{1,2}(A)$ with $\parallel \boldsymbol{\phi}_n \parallel_{1,2} = 1$ converging weakly to some $\tilde{\boldsymbol{\phi}}$ and such that

$$D^2E_0(\bar{\boldsymbol{u}}_0)(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\phi}}) \leq \liminf_{n\to\infty} D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n,\boldsymbol{\phi}_n) \tag{6.3.9}$$

holds where $D^2E_0(\bar{\boldsymbol{u}}_0)(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\phi}})$ is given by (6.3.7) and

$$D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n,\boldsymbol{\phi}_n) = \int_A |\nabla\boldsymbol{\phi}_n|^2$$
$$+ \frac{2}{\delta_n}h'(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))\det(\nabla\boldsymbol{\phi}_n) + \frac{1}{\delta_n}h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_{\delta_n})^T.\nabla\boldsymbol{\phi}_n)^2 \mathrm{~dx} \tag{6.3.10}$$

by showing that

$$\int_A |\nabla\tilde{\boldsymbol{\phi}}|^2 \mathrm{~dx} \leq \liminf_{n\to\infty} \int_A |\nabla\boldsymbol{\phi}_n|^2 \mathrm{~dx} \tag{6.3.11}$$

and

$$\int_A h''(\det(\nabla\bar{\boldsymbol{u}}_0))((\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_0))^T.\nabla\tilde{\boldsymbol{\phi}})^2 \mathrm{~dx}$$
$$\leq \liminf_{n\to\infty} \int_A h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))((\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_{\delta_n}))^T.\nabla\boldsymbol{\phi}_n)^2 \mathrm{~dx} \tag{6.3.12}$$

hold and that

$$\int_A \frac{2}{\delta_n}h'(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))\det(\nabla\boldsymbol{\phi}_n) \mathrm{~dx} \tag{6.3.13}$$

converges to a limit as $n \to \infty$.
We will then show that $\tilde{\phi}$ must satisfy $(\mathrm{adj}(\nabla \bar{u}_0))^T . \nabla \tilde{\phi} = 0$ a.e., and from there that $\tilde{\phi} \equiv \mathbf{0}$ and that $\phi_n \to \tilde{\phi}$ in $W_0^{1,2}(A)$ to force a contradiction. We proceed in four steps.
Step 1. Consider a sequence (δ_n) such that $\delta_n \to 0$ as $n \to \infty$, and so there exists a sequence $(\phi_n) \in W_0^{1,2}(A)$ $(\phi_n \neq 0)$ such that $D^2 E_{\delta_n}(\bar{u}_{\delta_n})(\phi_n, \phi_n) \leq \delta_n \parallel \phi_n \parallel_{1,2}^2$. By replacing ϕ_n with $\dfrac{\phi_n}{\parallel \phi_n \parallel_{1,2}}$ we may suppose, without loss of generality, that $\parallel \phi_n \parallel_{1,2} = 1$ for all $n \in \mathbb{N}$. Thus $D^2 E_{\delta_n}(\bar{u}_{\delta_n})(\phi_n, \phi_n) \leq \delta_n$ and (ϕ_n) is bounded in $W_0^{1,2}(A)$ and so contains a weakly convergent subsequence, still labelled (ϕ_n), converging weakly to some $\tilde{\phi} \in W_0^{1,2}(A)$. Now, by sequential weak lower semicontinuity of the norm on $W_0^{1,2}$, (6.3.11) holds.
Step 2. We now show that the term (6.3.13) converges to a limit as $n \to \infty$.
By hypothesis (S2) and since

$$\int_A h'(\det(\nabla \bar{u}_0)) \det(\nabla \phi_n) \ \mathrm{dx} = 0$$

we can write (6.3.13) as

$$\begin{aligned}
&\int_A \frac{2}{\delta_n} h'(\det(\nabla \bar{u}_{\delta_n})) \det(\nabla \phi_n) \ \mathrm{dx} \\
&\quad = \int_A 2[h''(\det(\nabla \bar{u}_0))(\mathrm{adj}(\nabla \bar{u}_0)^T . \nabla w_1) \\
&\qquad + \delta_n h'''(\det(\nabla \bar{u}_0))(\mathrm{adj}(\nabla \bar{u}_0)^T . \nabla w_2 + \det(\nabla w_1)) + O(\delta_n^2)] \det(\nabla \phi_n) \ \mathrm{dx} \\
&\quad = \int_A 2 \left\{ A_0(x) + \delta_n A_1(x) + \delta_n^2 \hat{F}(x, \delta_n) \right\} \det(\nabla \phi_n) \ \mathrm{dx}.
\end{aligned}$$

Now, $\hat{F}(x, \delta_n)$ is bounded. Also A_0 is a function bounded in C^1, A_1 is a bounded L^∞ function and $\det(\nabla \phi_n)$ is bounded in L^1 as ϕ_n is bounded in $W^{1,2}$. Therefore, both $A_1(x) \det(\nabla \phi_n)$ and $\hat{F}(x, \delta_n) \det(\nabla \phi_n)$ are bounded in L^1, and so we have that

$$\int_A \left\{ \delta_n A_1(x) + \delta_n^2 \hat{F}(x, \delta_n) \right\} \det(\nabla \phi_n) \ \mathrm{dx} \to 0$$

as $n \to \infty$. Hence, we are left with the term

$$\int_A A_0(x) \det(\nabla \phi_n) \ \mathrm{dx}$$

where A_0 is a function bounded in C^1.
We write $\det(\nabla \phi_n) = \phi^1_{(n),1} \phi^2_{(n),2} - \phi^1_{(n),2} \phi^2_{(n),1}$. As ϕ_n, $\tilde{\phi}$ are in $W_0^{1,2}(A)$ it follows

that $\det(\nabla\boldsymbol{\phi}_n)$, $\det(\nabla\tilde{\boldsymbol{\phi}})$ are in $L^1(A)$. Now we follow the approach used in the proof of [7, Lemma 3.3]. As $A_0(\boldsymbol{x}) \in C^1(A)$, we can use the divergence theorem to write

$$\int_A A_0(\boldsymbol{x})(\phi^1_{(n),1}\phi^2_{(n),2} - \phi^1_{(n),2}\phi^2_{(n),1})\ \mathrm{dx}$$
$$= \int_A -(A_0(\boldsymbol{x}))_{,1}(\phi^1_{(n)}\phi^2_{(n),2}) + (A_0(\boldsymbol{x}))_{,2}(\phi^1_{(n)}\phi^2_{(n),1})\ \mathrm{dx}, \tag{6.3.14}$$

where the boundary terms vanish, since $\boldsymbol{\phi}_n = \mathbf{0}$ on ∂A. The equality (6.3.14) holds if $\tilde{\boldsymbol{\phi}} \in C^\infty(A)$ and $C^\infty(A)$ is norm dense in $W^{1,2}(A)$. Now as $\phi^i_{(n)} \rightharpoonup \tilde{\phi}^i$ in $W^{1,2}_0(A)$ $(i = 1, 2)$, then $\phi^i_{(n)} \to \tilde{\phi}^i$ in $L^2(A)$ $(i = 1, 2)$ by the Rellich-Kondrachov theorem, and so we have that $\phi^1_{(n)}\phi^2_{(n),k} \rightharpoonup \tilde{\phi}^1\tilde{\phi}^2_{,k}$ in L^1 $(k = 1, 2)$ by Proposition 3.1.20. As $A_0(\boldsymbol{x})$ is a function bounded in C^1, we can say that

$$\int_A -(A_0(\boldsymbol{x}))_{,1}(\phi^1_{(n)}\phi^2_{(n),2}) + (A_0(\boldsymbol{x}))_{,2}(\phi^1_{(n)}\phi^2_{(n),1})\ \mathrm{dx}$$
$$\to \int_A -(A_0(\boldsymbol{x}))_{,1}(\tilde{\phi}^1\tilde{\phi}^2_2) + (A_0(\boldsymbol{x}))_{,2}(\tilde{\phi}^1\tilde{\phi}^2_1)\ \mathrm{dx}$$

as $n \to \infty$. Now

$$\int_A -(A_0(\boldsymbol{x}))_{,1}(\tilde{\phi}^1\tilde{\phi}^2_2) + (A_0(\boldsymbol{x}))_{,2}(\tilde{\phi}^1\tilde{\phi}^2_1)\ \mathrm{dx} = \int_A A_0(\boldsymbol{x})(\tilde{\phi}^1_1\tilde{\phi}^2_2 - \tilde{\phi}^1_2\tilde{\phi}^2_1)\ \mathrm{dx}. \tag{6.3.15}$$

holds for all $\tilde{\boldsymbol{\phi}} \in W^{1,2}_0(A)$. Comparing (6.3.14) with (6.3.15) we find that

$$\int_A A_0(\boldsymbol{x})(\phi^1_{(n),1}\phi^2_{(n),2} - \phi^1_{(n),2}\phi^2_{(n),1})\ \mathrm{dx} \to \int_A A_0(\boldsymbol{x})(\tilde{\phi}^1_1\tilde{\phi}^2_2 - \tilde{\phi}^1_2\tilde{\phi}^2_1)\ \mathrm{dx}$$

as $n \to \infty$.
Also, $A_0(\boldsymbol{x}) = h''(\det(\nabla\bar{\boldsymbol{u}}_0))(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_0)^T.\nabla\boldsymbol{w}_1) = -\bar{P}(\boldsymbol{x})$ from [45]. Thus

$$\int_A \frac{2}{\delta_n}h'(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))\det(\nabla\boldsymbol{\phi}_n)\ \mathrm{dx} \to \int_A -2\bar{P}(\boldsymbol{x})\det(\nabla\tilde{\boldsymbol{\phi}})\ \mathrm{dx}. \tag{6.3.16}$$

Step 3. We now consider the term

$$\int_A \frac{1}{\delta_n}h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))((\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_{\delta_n}))^T.\nabla\boldsymbol{\phi}_n)^2\ \mathrm{dx}. \tag{6.3.17}$$

We know due to convexity of h that $h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n})) \geq 0$.
Put $V_{\delta_n} := h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))^{\frac{1}{2}}(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_{\delta_n}))^T$. Thus $V_0 = h''(\det(\nabla\bar{\boldsymbol{u}}_0))^{\frac{1}{2}}(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_0))^T$

and $V_{\delta_n} \to V_0$ as $n \to \infty$ uniformly in L^∞. Also $V_{\delta_n}.\nabla\boldsymbol{\phi}_n \rightharpoonup V_0.\nabla\tilde{\boldsymbol{\phi}}$ in L^2 as $\nabla\boldsymbol{\phi}_n \rightharpoonup \nabla\tilde{\boldsymbol{\phi}}$ in L^2. Thus we can say that

$$\int_A (V_{\delta_n}.\nabla\boldsymbol{\phi}_n)^2 \, \mathrm{dx}$$

is sequentially weakly lower semicontinuous and so

$$\int_A (V_0.\nabla\tilde{\boldsymbol{\phi}})^2 \, \mathrm{dx} \leq \liminf_{n\to\infty} \int_A (V_{\delta_n}.\nabla\boldsymbol{\phi}_n)^2 \, \mathrm{dx}. \tag{6.3.18}$$

Step 4. We return to consider $D^2E(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n)$. By (6.3.11), (6.3.16) and (6.3.18) we have that (6.3.9) holds. We now show that $\tilde{\boldsymbol{\phi}}$ must be such that $(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_0))^T.\nabla\tilde{\boldsymbol{\phi}} = 0$ a.e. holds.
Suppose (for a contradiction) that $(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_0))^T.\nabla\tilde{\boldsymbol{\phi}} \neq 0$ on a set of positive measure (and thus $\tilde{\boldsymbol{\phi}} \neq 0$). Then

$$l = \liminf_{n\to\infty} \int_A (V_{\delta_n}.\nabla\boldsymbol{\phi}_n)^2 \, \mathrm{dx} \geq \int_A (V_0.\nabla\tilde{\boldsymbol{\phi}})^2 \, \mathrm{dx} = m > 0.$$

Hence

$$\liminf_{n\to\infty} \frac{1}{\delta_n} \int_A (V_{\delta_n}.\nabla\boldsymbol{\phi}_n)^2 \, \mathrm{dx} = \infty \tag{6.3.19}$$

and so by (6.3.11), (6.3.16) and (6.3.19) we have that

$$\liminf_{n\to\infty} D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) = \infty$$

- a contradiction, since we must have $\liminf_{n\to\infty} D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) \leq \delta_n < \bar{\delta}$ as for each n we are assuming that $D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) \leq \delta_n < \bar{\delta}$. Hence $\tilde{\boldsymbol{\phi}}$ must be such that $(\mathrm{adj}(\nabla\bar{\boldsymbol{u}}_0))^T.\nabla\tilde{\boldsymbol{\phi}} = 0$ a.e..
Now, as we made the assumption that $D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) \leq \delta_n$, then

$$\liminf_{n\to\infty} D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) \leq 0.$$

Hence, $D^2E_0(\bar{\boldsymbol{u}}_0)(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}) \leq 0$. However $D^2E_0(\bar{\boldsymbol{u}}_0)(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}) \not< 0$ and so we have

$$D^2E_0(\bar{\boldsymbol{u}}_0)(\tilde{\boldsymbol{\phi}}, \tilde{\boldsymbol{\phi}}) = \liminf_{n\to\infty} D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n, \boldsymbol{\phi}_n) = 0,$$

and thus $\tilde{\boldsymbol{\phi}} \equiv \mathbf{0}$. Hence, there exists a subsequence, still labelled $(\boldsymbol{\phi}_n)$, that must satisfy $\lim_{n\to\infty} D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n,\boldsymbol{\phi}_n)=0$. Thus

$$D^2E_{\delta_n}(\bar{\boldsymbol{u}}_{\delta_n})(\boldsymbol{\phi}_n,\boldsymbol{\phi}_n)\to D^2E_0(\bar{\boldsymbol{u}}_0)(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\phi}})=0. \tag{6.3.20}$$

Due to (6.3.10), (6.3.16) and (6.3.20) we can say that

$$\int_A |\nabla\boldsymbol{\phi}_n|^2+\frac{1}{\delta_n}h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))(\operatorname{adj}(\nabla\bar{\boldsymbol{u}}_{\delta_n})^T.\nabla\boldsymbol{\phi}_n)^2\,\mathrm{dx}\to\int_A|\nabla\tilde{\boldsymbol{\phi}}|^2\,\mathrm{dx}. \tag{6.3.21}$$

Now, we recall that $\boldsymbol{\phi}_n \rightharpoonup \tilde{\boldsymbol{\phi}}$ in $W_0^{1,2}(A)$. Also, note that from (6.3.21) we have $\lim_{n\to\infty} \| \boldsymbol{\phi}_n \|_{1,2}\leq\| \tilde{\boldsymbol{\phi}} \|_{1,2}$ as

$$\int_A\frac{1}{\delta_n}h''(\det(\nabla\bar{\boldsymbol{u}}_{\delta_n}))(\operatorname{adj}(\nabla\bar{\boldsymbol{u}}_{\delta_n})^T.\nabla\boldsymbol{\phi}_n)^2\,\mathrm{dx}\geq 0.$$

Also, as (6.3.11) holds, we have that $\| \tilde{\boldsymbol{\phi}} \|_{1,2}\leq \liminf_{n\to\infty} \| \boldsymbol{\phi}_n \|_{1,2}$. Thus $\| \tilde{\boldsymbol{\phi}} \|_{1,2}= \liminf_{n\to\infty} \| \boldsymbol{\phi}_n \|_{1,2}= 1$. Thus $\| \tilde{\boldsymbol{\phi}} \|_{1,2}= 1$ - a contradiction as we assumed that $\tilde{\boldsymbol{\phi}}\equiv\mathbf{0}$. Hence, there exists a value of δ, $\bar{\delta}$ say, such that for all $\delta\in(0,\bar{\delta})$ and $\boldsymbol{\phi}\in W_0^{1,2}(A)$ we have $\delta^2E_\delta(\bar{\boldsymbol{u}}_\delta)(\boldsymbol{\phi},\boldsymbol{\phi})$ is $W^{1,2}$-positive definite. □

Remark 6.3.3 *We note that the identity is not a natural state when we consider the energy functional*

$$E_\delta(\boldsymbol{u}_\delta)=\int_A\frac{1}{2}|\nabla\boldsymbol{u}_\delta|^2+\frac{1}{\delta}h(\det(\nabla\boldsymbol{u}_\delta))\mathrm{dx}, \tag{6.3.22}$$

since on writing $|\nabla\boldsymbol{u}_\delta|^2=v_1^2+v_2^2$ *and* $\det(\nabla\boldsymbol{u}_\delta)=v_1v_2$ *(where* v_i *are the principal stretches), and putting* $v_i=1$*, we find that* $1+\dfrac{h'(1)}{\delta}\neq 0$ *if we assume that* $h'(1)=0$*, and hence the identity is not a natural state.*
We note that, for the case of cavitation, there are examples in [63] of stored energy functions satisfying the condition that the identity be a natural state by the introduction of a null Lagrangian.
With this in mind, we consider the functional

$$\hat{E}_\delta(\boldsymbol{u}_\delta)=\int_A\frac{1}{2}|\nabla\boldsymbol{u}_\delta|^2+\frac{1}{\delta}\left\{h(\det(\nabla\boldsymbol{u}_\delta))-\delta\det(\nabla\boldsymbol{u}_\delta)\right\}\mathrm{dx}. \tag{6.3.23}$$

In this case, on writing $|\nabla\boldsymbol{u}_\delta|^2=v_1^2+v_2^2$ *and* $\det(\nabla\boldsymbol{u}_\delta)=v_1v_2$*, and putting* $v_i=1$ *(with the assumption that* $h'(1)=0$*), we find that* $1+\dfrac{h'(1)}{\delta}-1=0$*, and hence the identity is a natural state. We note that* $\det(\nabla\boldsymbol{u}_\delta)$ *is a null Lagrangian. In fact,* *as we are mapping from an annulus to itself, all deformations $\boldsymbol{u}_\delta$ must satisfy*

$$\int_A \det(\nabla \boldsymbol{u}_\delta)\mathrm{dx} = \pi(b^2 - a^2) = C, \qquad (6.3.24)$$

and thus $\hat{E}_\delta(\boldsymbol{u}_\delta) = E_\delta(\boldsymbol{u}_\delta) + C$. Also, the first and second variations of $\hat{E}_\delta(\boldsymbol{u}_\delta)$ about the equilibrium solutions $\bar{\mathbf{u}}_\delta$ are the same as those of $E_\delta(\boldsymbol{u}_\delta)$ about the equilibrium solutions $\bar{\mathbf{u}}_\delta$. Hence, Proposition 6.3.2 holds when the total energy is of the form (6.3.23), with the additional fact that the identity is now a natural state.

Chapter 7

Symmetrisation arguments for deformations of a compressible annulus.

In the previous chapter rotationally symmetric equilibrium solutions for an annulus were shown to have minimising properties in general classes of deformations for two special cases: (possibly) degenerate deformations of a compressible annulus (§6.1) and deformations of an incompressible annulus (§6.2). In both these cases the corresponding energy functional considered was the Dirichlet integral. We also recall that [57, §4] proves the existence of equilibrium solutions in the class of rotationally symmetric deformations. There, the deformations are of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \rho(R)\begin{pmatrix} \cos(\theta+\psi(R)) \\ \sin(\theta+\psi(R)) \end{pmatrix}, \tag{7.0.1}$$

and the stored energy function is of the form

$$W(F) = \frac{1}{2}|F|^2 + h(\det(F)),$$

where, as mentioned previously, h is C^2, convex and such that $h(d) \to \infty$ as $d \to 0, \infty$.

We now turn our attention to consider, for the stored energy function that was originally considered in [57, §4], non-rotationally symmetric deformations $\tilde{\boldsymbol{u}}$:

$\bar{A} \to \bar{A}$ of the annulus A given by (4.0.1). For this we will put $\tilde{\boldsymbol{u}}$ to be of the form

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \tilde{\rho}(R,\theta) \begin{pmatrix} \cos(\theta + \tilde{\psi}(R,\theta)) \\ \sin(\theta + \tilde{\psi}(R,\theta)) \end{pmatrix}, \tag{7.0.2}$$

with $\tilde{\rho}(R,\theta)$ and $\tilde{\psi}(R,\theta)$ satisfying the following conditions (throughout this chapter):

(G1) $\tilde{\rho}(R,0) = \tilde{\rho}(R,2\pi)$ and $\tilde{\psi}(R,0) = \tilde{\psi}(R,2\pi)$ for all $R \in [a,b]$;

(G2) $\tilde{\rho}(a,\theta) = a$, $\tilde{\rho}(b,\theta) = b$ and $\tilde{\psi}(a,\theta) = 0$, $\tilde{\psi}(b,\theta) = 2N\pi$ for all $\theta \in [0,2\pi)$.

We consider these non-rotationally symmetric deformations with a view to investigating what can be said about minimisers in this class of deformations, in particular whether they are necessarily rotationally symmetric. It has already been shown that there exist minimisers in this class of general deformations, as indicated by [57, Theorem 3.1]. Hence, we consider an approach to the problem of showing whether the minimisers whose existence is shown in [57, Theorem 3.1] coincide with the rotationally symmetric minimisers whose existence is shown in [57, §4]. We will do this by investigating whether for any non-rotationally symmetric map there is a corresponding rotationally symmetric map that necessarily has less energy, where the two maps are related by a suitable constraint. The constraint we will introduce is an area constraint.
Throughout this chapter we let $\tilde{\boldsymbol{u}} : \bar{A} \to \bar{A}$ be a general map from the annulus to itself and $\boldsymbol{u} : \bar{A} \to \bar{A}$ will denote a corresponding symmetrised map of the form (7.0.1) (with both maps leaving the boundary as it is, that is $\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{x}$ for all $\boldsymbol{x} \in \partial A$).

Definition 7.0.1 *The map $\boldsymbol{u}$ is an* ***area preserving symmetrisation*** *of $\tilde{\boldsymbol{u}}$ if for each R in $[a,b]$ the area enclosed by the curve $\boldsymbol{u}(\boldsymbol{x})|_{|\boldsymbol{x}|=R}$ and the inner boundary of the annulus is equal to the area enclosed by the curve $\tilde{\boldsymbol{u}}(\boldsymbol{x})|_{|\boldsymbol{x}|=R}$ and the inner boundary (see Figure 7.1).*

With this in mind we let $B_R \setminus B_a$ be the annulus with inner radius a and outer radius R. The deformed area of $B_R \setminus B_a$ under the map $\boldsymbol{u}(\boldsymbol{x})$ is given by

$$\int_{B_R \setminus B_a} \det(\nabla \boldsymbol{u}) \mathrm{dx}.$$

As we are considering $\boldsymbol{u}$ of the form (7.0.1) we can give a mathematical representation of an area preserving symmetrisation.

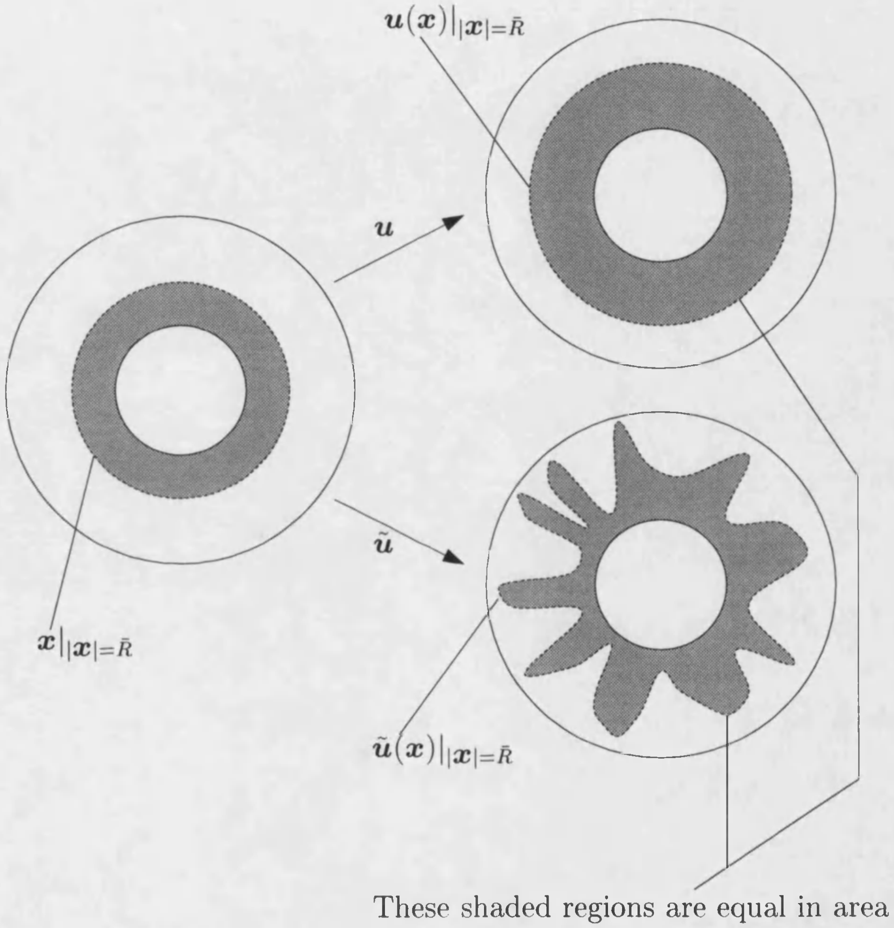

Figure 7.1: Two possible deformations $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$.

Lemma 7.0.2 *Let $\tilde{\boldsymbol{u}} : \bar{A} \to \bar{A}$ be a general map from the annulus to itself and let $\boldsymbol{u} : \bar{A} \to \bar{A}$ be a corresponding area preserving symmetrisation of the form (7.0.1) of $\tilde{\boldsymbol{u}}$, both satisfying identity boundary conditions. Then*

$$\int_{B_R \setminus B_a} \det(\nabla \tilde{\boldsymbol{u}}) \mathrm{dx} = \int_{B_R \setminus B_a} \det(\nabla \boldsymbol{u}) \mathrm{dx} = \pi((\rho(R))^2 - a^2) \qquad (7.0.3)$$

holds for each R in $[a, b]$.

Remark 7.0.3 *If $\boldsymbol{u}$ is an area preserving symmetrisation of $\tilde{\boldsymbol{u}}$, then the relation (7.0.3) now defines $\rho(R)$, and it follows from the differentiation of (7.0.3) with respect to R that*

$$\frac{\int_{S_R} \det(\nabla \tilde{\boldsymbol{u}})\ \mathrm{dS}}{2\pi R} = \det(\nabla \boldsymbol{u}) = \frac{\rho(R)}{R} \rho'(R), \qquad (7.0.4)$$

for R in $[a, b]$, where S_R is a circle of radius R. The right-hand equality follows from (4.0.8). Note that we are still free to choose $\psi(R)$ in the definition of $\boldsymbol{u}$ in (7.0.1).

7.1 Reduction to the Dirichlet integral.

We first show that the problem of whether the corresponding rotationally symmetric map $\boldsymbol{u}$ necessarily has less energy than $\tilde{\boldsymbol{u}}$ can be reduced to investigating the effect of symmetrisation on the Dirichlet integral. Define $\hat{E}(\tilde{\boldsymbol{u}})$ as

$$\hat{E}(\tilde{\boldsymbol{u}}) := \frac{1}{2} \int_{R=a}^{\bar{R}} \int_{S_R} |\nabla \tilde{\boldsymbol{u}}|^2\ dR\ \mathrm{dS} + \int_{R=a}^{\bar{R}} \int_{S_R} h(\det(\nabla \tilde{\boldsymbol{u}}))\ dR\ \mathrm{dS}. \qquad (7.1.1)$$

We demonstrate that the area preserving symmetrisation $\boldsymbol{u}$ reduces the term

$$\int_{B_{\bar{R}} \setminus B_a} h(\det(\nabla \tilde{\boldsymbol{u}}))\ \mathrm{dx}, \qquad (7.1.2)$$

where $B_{\bar{R}} \setminus B_a$ is the annulus with outer radius $\bar{R}$ and inner radius a. In showing this we make use of Jensen's inequality.

Theorem 7.1.1 (Jensen's inequality) *Let $\Omega \in \mathbb{R}^n$ be a bounded open set, $u \in L^1(\Omega)$ and $f : \mathbb{R} \to \mathbb{R}$ be convex. Then*

$$f\left(\frac{1}{\mathrm{meas}(\Omega)} \int_{\Omega} u(x)\ \mathrm{dx}\right) \leq \frac{1}{\mathrm{meas}(\Omega)} \int_{\Omega} f(u(x))\ \mathrm{dx}.$$

We now show that (7.1.2) is lowered by an area preserving symmetrisation.

Proposition 7.1.2 *Let* $\boldsymbol{u}$ *be an area preserving symmetrisation of* $\tilde{\boldsymbol{u}}$ *and let* h *be convex. Then*

$$\int_{B_R \backslash B_a} h(\det(\nabla \tilde{\boldsymbol{u}}))\mathrm{dx} \geq \int_{B_R \backslash B_a} h(\det(\nabla \boldsymbol{u}))\mathrm{dx}.$$

Proof: Since h is convex, by using Jensen's inequality we find that

$$\begin{aligned} \int_{S_R} h(\det(\nabla \tilde{\boldsymbol{u}}))\ \mathrm{dS} &\geq 2\pi R h\left(\frac{1}{2\pi R}\int_{S_R} \det(\nabla \tilde{\boldsymbol{u}})\ \mathrm{dS}\right) \\ &= 2\pi R h\left(\frac{\rho(R)}{R}\rho'(R)\right) = \int_{S_R} h(\det(\nabla \boldsymbol{u}))\ \mathrm{dS}, \end{aligned} \tag{7.1.3}$$

using (7.0.4). Hence

$$\int_{B_R \backslash B_a} h(\det(\nabla \tilde{\boldsymbol{u}}))\ \mathrm{dx} \geq \int_{B_R \backslash B_a} h(\det(\nabla \boldsymbol{u}))\ \mathrm{dx}$$

as required. □

Remark 7.1.3 *Thus it follows that if* $\boldsymbol{u}$ *is an area preserving symmetrisation*

$$\hat{E}(\tilde{\boldsymbol{u}}) \geq \frac{1}{2}\int_{R=a}^{\bar{R}}\int_{S_R} |\nabla \tilde{\boldsymbol{u}}|^2\ dR\ \mathrm{dS} + 2\pi \int_{R=a}^{\bar{R}} Rh\left(\frac{\rho(R)}{R}\rho'(R)\right) dR,$$

so that in particular

$$\begin{aligned} E(\tilde{\boldsymbol{u}}) \geq & \frac{1}{2}\int_{R=a}^{b}\int_{S_R} |\nabla \tilde{\boldsymbol{u}}|^2\ dR\ \mathrm{dS} + 2\pi \int_{R=a}^{b} Rh\left(\frac{\rho(R)}{R}\rho'(R)\right) dR \\ &= \int_A \frac{1}{2}|\nabla \tilde{\boldsymbol{u}}|^2 + h(\det(\nabla \boldsymbol{u})\ \mathrm{dx}. \end{aligned} \tag{7.1.4}$$

7.2 Investigation of the Dirichlet integral.

We now investigate the effect of the area preserving symmetrisation $\boldsymbol{u}$ on the Dirichlet integral. We remark that it is still an open question as to what is the best symmetrisation of $\tilde{\boldsymbol{u}}$ to consider, bearing in mind that we are still free to choose $\psi(R)$ in the definition of an area preserving symmetrisation, and whether the Dirichlet integral is lowered on considering such a symmetrisation. However, we can simplify the problem further.

7.2.1 A further reduction.

If we write

$$\frac{1}{2}\int_A |\nabla \tilde{\boldsymbol{u}}|^2 \mathrm{dx} = \frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 + \frac{1}{R^2}\left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}\right|^2 \mathrm{dx} \tag{7.2.1}$$

then we can show that the integral of the square of the tangential component

$$\frac{1}{2}\int_A \frac{1}{R^2}\left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}\right|^2 \mathrm{dx} \tag{7.2.2}$$

is reduced by an area preserving symmetrisation, so that the problem further reduces to considering the integral of the square of the radial component

$$\frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 \mathrm{dx}. \tag{7.2.3}$$

In order to show this we note that for $\boldsymbol{u}$ of the form (7.0.1) we have

$$\frac{1}{2}\int_A |\nabla \boldsymbol{u}|^2 \mathrm{dx} = \pi \int_{R=a}^{b} \left[R\left(\frac{\partial \rho}{\partial R}(R)\right)^2 + R\rho^2(R)\left(\frac{\partial \psi}{\partial R}(R)\right)^2\right] + \left\{\frac{\rho^2(R)}{R}\right\} dR, \tag{7.2.4}$$

and thus

$$\frac{1}{2}\int_A \frac{1}{R^2}\left|\frac{\partial \boldsymbol{u}}{\partial \theta}\right|^2 \mathrm{dx} = \pi \int_{R=a}^{b} \frac{\rho^2(R)}{R}\, dR. \tag{7.2.5}$$

We also note that for fixed R, $R = \bar{R}$ say, the length of the closed curve $\tilde{\boldsymbol{u}}(\boldsymbol{x})|_{|\boldsymbol{x}|=\bar{R}}$ is

$$\int_{\theta=0}^{2\pi} \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}\right| d\theta.$$

We now give the result.

Proposition 7.2.1 *Let $\boldsymbol{u}$ be an area preserving symmetrisation of $\tilde{\boldsymbol{u}}$. Then*

$$\int_{\theta=0}^{2\pi} \frac{1}{R}\left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}\right|^2 d\theta \geq 2\pi \frac{\rho^2(R)}{R} = \int_{\theta=0}^{2\pi} \frac{1}{R}\left|\frac{\partial \boldsymbol{u}}{\partial \theta}\right|^2 d\theta.$$

Proof: By Jensen's inequality we have

$$\int_{\theta=0}^{2\pi} \frac{1}{R} \left| \frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta} \right|^2 d\theta \geq \frac{2\pi}{R} \left(\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left| \frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta} \right| d\theta \right)^2 . \tag{7.2.6}$$

Now the curve of smallest length that encloses a fixed area is a circle (by the classical isoperimetric inequality). Also, from (7.0.3) the area enclosed between the curve $\tilde{\boldsymbol{u}}(\boldsymbol{x})|_{|\boldsymbol{x}|=R}$ and the inner boundary of A is equal to the area enclosed between the curve $\boldsymbol{u}(\boldsymbol{x})|_{|\boldsymbol{x}|=R}$ and the inner boundary. Thus by the isoperimetric inequality $\boldsymbol{u}(\boldsymbol{x})|_{|\boldsymbol{x}|=R}$ is a circle of radius $\rho(R)$. Thus we must have that

$$\int_{\theta=0}^{2\pi} \left| \frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta} \right| d\theta \geq 2\pi\rho(R), \tag{7.2.7}$$

and so by (7.2.5), (7.2.6) and (7.2.7)

$$\int_{\theta=0}^{2\pi} \frac{1}{R} \left| \frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta} \right|^2 d\theta \geq \frac{2\pi}{R}\rho^2(R) = \int_{\theta=0}^{2\pi} \frac{1}{R} \left| \frac{\partial \boldsymbol{u}}{\partial \theta} \right|^2 d\theta$$

as required. □

7.2.2 Explicit form of the Dirichlet integral and the area preserving symmetrisation.

Up until now we have not assumed a form for $\tilde{\boldsymbol{u}}$ in proving the above results. In order to progress we consider maps $\tilde{\boldsymbol{u}}$ of the form (7.0.2) with $\tilde{\rho}(R,\theta)$ and $\tilde{\psi}(R,\theta)$ satisfying (G1) and (G2). The next result shows how the Dirichlet integral, $\det(\nabla\tilde{\boldsymbol{u}})$ and (7.0.4) can be written when $\tilde{\boldsymbol{u}}$ is given by (7.0.2).

Lemma 7.2.2 *Let $\tilde{\boldsymbol{u}}$ be given by (7.0.2). Then*

$$\det(\nabla\tilde{\boldsymbol{u}}) = \frac{1}{R}\left(\frac{\partial\tilde{\rho}}{\partial R}(R,\theta)\tilde{\rho}(R,\theta)\left(1+\frac{\partial\tilde{\psi}}{\partial\theta}(R,\theta)\right) - \tilde{\rho}(R,\theta)\frac{\partial\tilde{\rho}}{\partial\theta}(R,\theta)\frac{\partial\tilde{\psi}}{\partial R}(R,\theta)\right), \tag{7.2.8}$$

and

$$\frac{1}{2}\int_A |\nabla\tilde{\boldsymbol{u}}|^2 \,\mathrm{dx}$$

$$= \frac{1}{2}\int_{R=a}^{b}\left[R\int_{\theta=0}^{2\pi}\left|\frac{\partial\tilde{\rho}}{\partial R}(R,\theta)\right|^2 d\theta + R\int_{\theta=0}^{2\pi}\tilde{\rho}^2(R,\theta)\left|\frac{\partial\tilde{\psi}}{\partial R}(R,\theta)\right|^2 d\theta\right]$$

$$+\left\{\frac{1}{R}\int_{\theta=0}^{2\pi}\tilde{\rho}^2(R,\theta)\left|1+\frac{\partial\tilde{\psi}}{\partial\theta}(R,\theta)\right|^2 d\theta + \frac{1}{R}\int_{\theta=0}^{2\pi}\left|\frac{\partial\tilde{\rho}}{\partial\theta}(R,\theta)\right|^2 d\theta\right\} dR. \tag{7.2.9}$$

Further, from (7.0.4) we get

$$\pi\rho^2(R) = \int_{\theta=0}^{2\pi}\frac{1}{2}\tilde{\rho}^2(R,\theta)\frac{\partial}{\partial\theta}(\theta+\tilde{\psi}(R,\theta))d\theta. \tag{7.2.10}$$

This is straightforward to show on noting that

$$\nabla\tilde{\boldsymbol{u}} = \begin{pmatrix} \dfrac{\partial\tilde{\rho}}{\partial R}\cos(\theta+\tilde{\psi}) - \tilde{\rho}\dfrac{\partial\tilde{\psi}}{\partial R}\sin(\theta+\tilde{\psi}) \\ \\ \dfrac{\partial\tilde{\rho}}{\partial R}\sin(\theta+\tilde{\psi}) + \tilde{\rho}\dfrac{\partial\tilde{\psi}}{\partial R}\cos(\theta+\tilde{\psi}) \end{pmatrix} \otimes \frac{\boldsymbol{x}}{R}$$

$$+\frac{1}{R}\begin{pmatrix} \dfrac{\partial\tilde{\rho}}{\partial\theta}\cos(\theta+\tilde{\psi}) - \tilde{\rho}\left(1+\dfrac{\partial\tilde{\psi}}{\partial\theta}\right)\sin(\theta+\tilde{\psi}) \\ \\ \dfrac{\partial\tilde{\rho}}{\partial\theta}\sin(\theta+\tilde{\psi}) + \tilde{\rho}\left(1+\dfrac{\partial\tilde{\psi}}{\partial\theta}\right)\cos(\theta+\tilde{\psi}) \end{pmatrix} \otimes \frac{1}{R}\begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}. \tag{7.2.11}$$

We now consider the problem of symmetrising $\tilde{\boldsymbol{u}}$. This is difficult to do in general, and so in §7.3 we will consider two special cases. The first case will be when $\tilde{\rho}(R,\theta) = \rho(R)$, and the second will be when $\tilde{\psi}(R,\theta) = \psi(R)$.

7.3 Two special cases.

7.3.1 $\tilde{\rho}(R, \theta) = \rho(R)$.

Let $\tilde{\boldsymbol{u}}$ be of the form

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \rho(R) \begin{pmatrix} \cos(\theta + \tilde{\psi}(R,\theta)) \\ \sin(\theta + \tilde{\psi}(R,\theta)) \end{pmatrix}. \tag{7.3.1}$$

Now

$$\frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 \mathrm{dx} = \frac{1}{2}\int_{R=a}^{b} 2\pi R \left|\frac{\partial \rho}{\partial R}(R)\right|^2 + R\rho^2(R)\left(\int_{\theta=0}^{2\pi}\left|\frac{\partial \tilde{\psi}}{\partial R}(R,\theta)\right|^2 d\theta\right) dR. \tag{7.3.2}$$

We now show that we can choose a specific form for ψ in the area preserving symmetrisation that lowers the Dirichlet integral.

Proposition 7.3.1 *Let $\tilde{\boldsymbol{u}}$ be of the form (7.3.1) and $\boldsymbol{u}$ be an area preserving symmetrisation of the form (7.0.1). Define*

$$\psi(R) := \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \tilde{\psi}(R,\theta) d\theta. \tag{7.3.3}$$

Then $\psi(R)$ satisfies (G2) *and*

$$\frac{1}{2}\int_A \left|\frac{\partial \boldsymbol{u}}{\partial R}\right|^2 \mathrm{dx} \le \frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 \mathrm{dx}$$

and thus $\int_A |\nabla \boldsymbol{u}|^2 \,\mathrm{dx} \le \int_A |\nabla \tilde{\boldsymbol{u}}|^2 \,\mathrm{dx}$.

Proof: If we consider the symmetrisation of $\tilde{\psi}(R,\theta)$ as given by (7.3.3), then $\psi(a) = \tilde{\psi}(a,\theta) = 0$ and $\psi(b) = \tilde{\psi}(b,\theta) = 2\pi$ for all $\theta \in [0, 2\pi)$ and

$$\frac{\partial \psi}{\partial R}(R) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{\partial \tilde{\psi}}{\partial R}(R,\theta) d\theta.$$

Thus by Jensen's inequality

$$\left(\frac{\partial \psi}{\partial R}(R)\right)^2 = \left(\frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{\partial \tilde{\psi}}{\partial R}(R,\theta) d\theta\right)^2 \le \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \left|\frac{\partial \tilde{\psi}}{\partial R}(R,\theta)\right|^2 d\theta, \tag{7.3.4}$$

and so by substituting (7.3.4) into (7.3.2) we have that

$$
\begin{aligned}
&\frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 \mathrm{dx} \\
&\geq \; 2\pi \int_{R=a}^{b} \frac{R}{2}\left[\left(\frac{\partial \rho}{\partial R}(R)\right)^2 + \rho^2(R)\left(\frac{\partial \psi}{\partial R}(R)\right)^2\right] dR = \frac{1}{2}\int_A \left|\frac{\partial \boldsymbol{u}}{\partial R}\right|^2 \mathrm{dx}
\end{aligned}
$$

as required. □

Hence the Dirichlet integral is lowered by the symmetrisation (7.3.3) by the arguments of §7.1 and §7.2.

7.3.2 $\tilde{\psi}(R,\theta) = \psi(R)$.

Let $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ be of the form

$$
\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \tilde{\rho}(R,\theta)\begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix}. \tag{7.3.5}
$$

Here we find that

$$
\begin{aligned}
&\frac{1}{2}\int_A |\nabla \tilde{\boldsymbol{u}}|^2 \mathrm{dx} = \\
&\quad \frac{1}{2}\int_{R=a}^{b}\left[R\int_{\theta=0}^{2\pi}\left|\frac{\partial \tilde{\rho}}{\partial R}(R,\theta)\right|^2 d\theta + R\left|\frac{\partial \psi}{\partial R}(R)\right|^2 \int_{\theta=0}^{2\pi}\tilde{\rho}^2(R,\theta)d\theta\right] \\
&\qquad + \left\{\frac{1}{R}\int_{\theta=0}^{2\pi}\tilde{\rho}^2(R,\theta)d\theta + \frac{1}{R}\int_{\theta=0}^{2\pi}\left|\frac{\partial \tilde{\rho}}{\partial \theta}(R,\theta)\right|^2 d\theta\right\} dR.
\end{aligned} \tag{7.3.6}
$$

By (7.2.10) it follows from the area preserving symmetrisation that

$$
\frac{\rho^2(R)}{2} = \frac{1}{2\pi}\int_{\theta=0}^{2\pi}\frac{1}{2}\tilde{\rho}^2(R,\theta)d\theta. \tag{7.3.7}
$$

However, this case contrasts with §7.3.1 in that it is not clear from (7.3.7) that the term

$$
\frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 \mathrm{dx}
$$

is lowered on symmetrisation of $\tilde{\boldsymbol{u}}$. Hence, another approach is required if we wish to investigate the effect of this symmetrisation on the Dirichlet integral.

In this case we will consider the difference in the Dirichlet integral ΔE between the rotationally symmetric map $\boldsymbol{u}$ (corresponding to $\rho(R)$) and the non-rotationally symmetric map $\tilde{\boldsymbol{u}}$ (corresponding to $\tilde{\rho}(R,\theta)$). Hence we consider

$$\Delta E := \frac{1}{2}\int_A |\nabla \tilde{\boldsymbol{u}}|^2 \,\mathrm{dx} - \frac{1}{2}\int_A |\nabla \boldsymbol{u}|^2 \,\mathrm{dx} \tag{7.3.8}$$

where $\boldsymbol{u}$ is given by (7.0.1). Our approach is to write the non-rotationally symmetric map in terms of a Fourier series expansion and to investigate the first and second variations of this difference around the rotationally symmetric map $\boldsymbol{u}$. This Fourier expansion approach has been used in the study of solutions of the Ginzburg-Landau equations ([47]). We write $\tilde{\rho}(R,\theta)$ in the form

$$\tilde{\rho}(R,\theta) = \rho(R) + \phi(R,\theta) \tag{7.3.9}$$

where

$$\phi(R,\theta) = \sum_{n=0}^{\infty} \alpha_n(R)\cos(n\theta) + \beta_n(R)\sin(n\theta). \tag{7.3.10}$$

In order to progress we require the following lemmas which are consequences of the area preserving symmetrisation.

Lemma 7.3.2 *Let $\tilde{\rho}$ be of the form (7.3.9) and suppose that (7.3.7) holds. Then*

$$\int_{\theta=0}^{2\pi} 2\rho(R)\phi(R,\theta) + \phi^2(R,\theta)d\theta = 0. \tag{7.3.11}$$

Lemma 7.3.3 *Let $\tilde{\rho}$ be of the form (7.3.9) and suppose that (7.3.7) holds and in addition ϕ is of the form (7.3.10). Then we can write*

$$\alpha_0(R) = -\rho(R) \pm \left\{ \rho^2(R) - \frac{1}{2}\sum_{n=1}^{\infty} \alpha_n^2(R) + \beta_n^2(R) \right\}^{\frac{1}{2}}. \tag{7.3.12}$$

We will now give a result showing a way of writing ΔE.

Proposition 7.3.4 *Let ΔE be given by (7.3.8) with $\boldsymbol{u}$ given by (7.0.1) and $\tilde{\boldsymbol{u}}$ by (7.3.5). Let $\tilde{\rho}$ be such that (7.3.7) and (7.3.9) hold, and let ϕ be of the form*

(7.3.10). Then ΔE can be written as

$$\Delta E = \frac{\pi}{2}\int_{R=a}^{b} R\left\{\left(2\rho^2 - \sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2)\right)^{-1}\left[\sum_{n=1}^{\infty} 2\left(\rho\frac{d\alpha_n}{dR} - \alpha_n\frac{d\rho}{dR}\right)^2 \right.\right.$$
$$+\sum_{n=1}^{\infty} 2\left(\rho\frac{d\beta_n}{dR} - \beta_n\frac{d\rho}{dR}\right)^2 - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\beta_m\frac{d\alpha_n}{dR} - \alpha_n\frac{d\beta_m}{dR}\right)^2$$
$$\left. - \left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\alpha_m\frac{d\alpha_n}{dR} - \alpha_n\frac{d\alpha_m}{dR}\right)^2 + \left(\beta_n\frac{d\beta_m}{dR} - \beta_m\frac{d\beta_n}{dR}\right)^2\right)\right]$$
$$\left. + \sum_{n=1}^{\infty}\left(\frac{n^2\alpha_n^2}{R^2} + \frac{n^2\beta_n^2}{R^2}\right)\right\} dR. \tag{7.3.13}$$

Proof: With $\tilde{\boldsymbol{u}}$ being of the form (7.3.5), where $\tilde{\rho}(R,\theta) = \rho(R) + \phi(R,\theta)$, and $\boldsymbol{u}$ of the form (7.0.1), the difference in the Dirichlet integral ΔE can be rewritten as

$$\Delta E = \frac{1}{2}\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{2\frac{d\rho}{dR}(R)\frac{\partial\phi}{\partial R}(R,\theta) + \left(\frac{\partial\phi}{\partial R}(R,\theta)\right)^2\right\} + \frac{1}{R}\left(\frac{\partial\phi}{\partial\theta}(R,\theta)\right)^2$$
$$+\left\{R\left(\frac{d\psi}{dR}(R)\right)^2 + \frac{1}{R}\right\}(2\rho(R)\phi(R,\theta) + \phi^2(R,\theta))\, dR\, d\theta. \tag{7.3.14}$$

By Lemma 7.3.2 we have

$$\Delta E =$$
$$\frac{1}{2}\int_{R=a}^{b}\int_{\theta=0}^{2\pi} R\left\{2\frac{d\rho}{dR}(R)\frac{\partial\phi}{\partial R}(R,\theta) + \left(\frac{\partial\phi}{\partial R}(R,\theta)\right)^2\right\} + \frac{1}{R}\left(\frac{\partial\phi}{\partial\theta}(R,\theta)\right)^2 dR\, d\theta. \tag{7.3.15}$$

Substitution of (7.3.10) into (7.3.15) results in ΔE being of the form

$$\Delta E = \pi\int_{R=a}^{b} R\left\{2\left(\frac{d\alpha_0}{dR}(R)\right)^2 + 4\frac{d\rho}{dR}(R)\frac{d\alpha_0}{dR}(R) + \sum_{n=1}^{\infty}\left(\frac{d\alpha_n}{dR}(R)\right)^2\right.$$
$$\left. + \left(\frac{d\beta_n}{dR}(R)\right)^2\right\} + \frac{1}{R}\sum_{n=1}^{\infty} n^2\alpha_n^2 + n^2\beta_n^2\, dR. \tag{7.3.16}$$

By Lemma 7.3.3 we can rewrite α_0 in the form (7.3.12). As a consequence of the boundary conditions $\rho(a) = a$, $\rho(b) = b$, $\alpha_n(a) = \alpha_n(b) = \beta_n(a) = \beta_n(b) = 0$ for $n = 0, 1, 2, \ldots$, we take the positive square root of (7.3.12). Now, putting

$$\Omega(R) := \frac{1}{2} \sum_{n=1}^{\infty} \alpha_n^2(R) + \beta_n^2(R)$$

and

$$\Gamma(R) := \Omega'(R) = \sum_{n=1}^{\infty} \frac{d\alpha_n}{dR}(R)\alpha_n(R) + \frac{d\beta_n}{dR}(R)\beta_n(R)$$

we note that

$$\alpha_0(R)(\alpha_0(R) + 2\rho(R)) = -\frac{1}{2} \sum_{n=1}^{\infty} \alpha_n^2(R) + \beta_n^2(R) = -\Omega(R)$$

and also that

$$\frac{d\alpha_0}{dR}(R) = -\frac{d\rho}{dR}(R) + \frac{1}{2(\rho^2(R) - \Omega(R))^{\frac{1}{2}}} \left(2\rho(R)\frac{d\rho}{dR}(R) - \Gamma(R) \right).$$

Thus, it can be shown that

$$\Delta E = \frac{\pi}{2} \int_a^b R \left\{ \sum_{n=1}^{\infty} \left(\frac{d\alpha_n}{dR}\right)^2 + \left(\frac{d\beta_n}{dR}\right)^2 + \frac{n^2}{R^2}(\alpha_n^2 + \beta_n^2) \right.$$
$$\left. - 2\left(\frac{d\rho}{dR}\right)^2 + \frac{(2\rho\rho' - \Gamma)^2}{2(\rho^2 - \Omega)} \right\} dR. \tag{7.3.17}$$

It then follows, after a lengthy but straightforward calculation, putting the integrand over a common denominator, to write ΔE as

$$\Delta E = \frac{\pi}{2}\int_{R=a}^{b} R\left\{\left(2\rho^2 - \sum_{n=1}^{\infty}(\alpha_n^2+\beta_n^2)\right)^{-1}\left[\sum_{n=1}^{\infty} 2\left(\rho\frac{d\alpha_n}{dR} - \alpha_n\frac{d\rho}{dR}\right)^2\right.\right.$$
$$+\sum_{n=1}^{\infty} 2\left(\rho\frac{d\beta_n}{dR} - \beta_n\frac{d\rho}{dR}\right)^2 - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\beta_m\frac{d\alpha_n}{dR} - \alpha_n\frac{d\beta_m}{dR}\right)^2$$
$$\left.-\left(\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\alpha_m\frac{d\alpha_n}{dR} - \alpha_n\frac{d\alpha_m}{dR}\right)^2 + \left(\beta_n\frac{d\beta_m}{dR} - \beta_m\frac{d\beta_n}{dR}\right)^2\right)\right]$$
$$\left.+\sum_{n=1}^{\infty}\left(\frac{n^2\alpha_n^2}{R^2} + \frac{n^2\beta_n^2}{R^2}\right)\right\} dR \tag{7.3.18}$$

as required. □

Remark 7.3.5 *If we consider (7.3.18), then on putting* $\alpha_n = \hat{\alpha}_n + \varepsilon\sigma_n$*,* $\beta_n = \hat{\beta}_n + \varepsilon\tau_n$*, where* $\hat{\alpha}_n = 0$ *and* $\hat{\beta}_n = 0$*,* $n = 1, 2, 3, \ldots$*, and considering the first and second variations of the energy difference* ΔE*, it is clear that the first variation* $(\Delta E)_{,\varepsilon}\big|_{\varepsilon=0} = 0$ *and it is straightforward to show from (7.3.18) that the second variation* $(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0}$ *has the form*

$$(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0} =$$
$$\pi\int_a^b \sum_{n=1}^{\infty} \frac{n^2}{R^2}(\sigma_n^2+\tau_n^2) + \frac{R}{\rho^2}\left\{\left(\rho\frac{d\sigma_n}{dR} - \sigma_n\frac{d\rho}{dR}\right)^2 + \left(\rho\frac{d\tau_n}{dR} - \tau_n\frac{d\rho}{dR}\right)^2\right\} dR \tag{7.3.19}$$

and thus

$$(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0} \geq \sum_{n=1}^{\infty} C_n\|\sigma_n\|_2^2 + D_n\|\tau_n\|_2^2 \geq 0.$$

We note that $(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0} > 0$ if $\sigma_n \not\equiv 0$ or $\tau_n \not\equiv 0$ for some $n \in \mathbb{N}$, and that (7.3.19) can be written in the form

$$(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0} =$$
$$\pi\int_a^b \sum_{n=1}^{\infty} A(R)((\sigma_n')^2 + (\tau_n')^2) + B(R)(\sigma_n\sigma_n' + \tau_n\tau_n') + C_n(R)(\sigma_n^2+\tau_n^2)\, dR \tag{7.3.20}$$

with $A(R) = R > 0$. It can be shown that

$$(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0} \geq \sum_{n=1}^{\infty} C_n' \|\sigma_n\|_{1,2}^2 + D_n' \|\tau_n\|_{1,2}^2 \geq 0.$$

In order to show this we require the following preparatory result.

Proposition 7.3.6 *Consider*

$$\mathcal{J}(\varphi) = \int_a^b P(R)(\varphi'(R))^2 + Q(R)\varphi(R)\varphi'(R) + S(R)\varphi^2(R)dR$$

on $W_0^{1,2}((a,b))$. Assume that P, Q, $S \in L^\infty((a,b))$, with $\zeta > P(R) \geq \delta > 0$ for all R. Then, if $\mathcal{J}(\varphi) > 0$ for all $\varphi \not\equiv 0$ in $W^{1,2}((a,b))$ then $\mathcal{J}(\varphi) \geq D\|\varphi\|_{1,2}^2$ for all φ in $W^{1,2}((a,b))$, where $D > 0$ is a constant.

Proof: Suppose (for a contradiction) that there exists a sequence $(\varphi_n) \subset W^{1,2}((a,b))$ ($\varphi_n \not\equiv 0$ for all $n \in \mathbb{N}$) such that $\mathcal{J}(\varphi_n) \leq \frac{1}{n} \| \varphi_n \|_{1,2}^2$. By replacing φ_n by $\dfrac{\varphi_n}{\| \varphi_n \|_{1,2}}$ if necessary we may suppose without loss of generality that

$$\| \varphi_n \|_{1,2}^2 = 1 \text{ for all } n \in \mathbb{N}.$$

Hence (φ_n) is bounded in $W^{1,2}$ and so contains a weakly convergent subsequence, still labelled (φ_n), converging weakly in $W^{1,2}$ to some $\tilde{\varphi} \in W^{1,2}$. Now, standard lower semicontinuity results imply that $\mathcal{J}(\tilde{\varphi}) \leq \liminf_{n\to\infty} \mathcal{J}(\varphi_n) = 0$ and so we have $\mathcal{J}(\tilde{\varphi}) = 0$ and thus $\tilde{\varphi} \equiv 0$. Hence, there exists a subsequence, still labelled (φ_n), such that $\mathcal{J}(\varphi_n) \to \mathcal{J}(\tilde{\varphi}) = 0$ as $n \to \infty$. Now, we have

$$\mathcal{J}(\varphi) = \int_a^b P(R)(\varphi'(R))^2 + Q(R)\varphi(R)\varphi'(R) + S(R)\varphi^2(R)\ dR,$$

where P, Q, $S \in L^\infty((a,b))$, $\zeta > P(R) \geq \delta > 0$ for all R. As $\varphi_n \rightharpoonup \tilde{\varphi}$ in $W^{1,2}((a,b))$ then by the Rellich-Kondrachov theorem $\varphi_n \to \tilde{\varphi}$ in $L^2((a,b))$. Also, as $\varphi_n' \rightharpoonup \tilde{\varphi}'$ in $L^2((a,b))$ we have that $\varphi_n\varphi_n' \rightharpoonup \tilde{\varphi}\tilde{\varphi}'$ in $L^1((a,b))$ by Proposition 3.1.20. Thus we have that

$$\int_a^b S(R)\varphi_n^2(R)dR \to \int_a^b S(R)\tilde{\varphi}^2(R)dR,$$

$$\int_a^b Q(R)\varphi_n(R)\varphi_n'(R)dR \to \int_a^b Q(R)\tilde{\varphi}(R)\tilde{\varphi}'(R)dR$$

as $n \to \infty$, and as $\mathcal{J}(\varphi_n) \to \mathcal{J}(\tilde{\varphi}) = 0$ as $n \to \infty$ we must have

$$\int_a^b P(R)(\varphi_n'(R))^2 dR \to \int_a^b P(R)(\tilde{\varphi}'(R))^2 dR,$$

and thus as $P(R)$ is in $L^\infty((a,b))$ and is bounded away from zero, as as (φ_n) is such that $\varphi_n \to \tilde{\varphi}$ in L^2 and $\varphi_n' \rightharpoonup \tilde{\varphi}'$ in L^2 it can be shown that

$$\int_a^b (\varphi_n'(R))^2 dR \to \int_a^b (\tilde{\varphi}'(R))^2 dR$$

as $n \to \infty$, and thus $\| \varphi_n \|_{1,2} \to \| \tilde{\varphi} \|_{1,2}$ as $n \to \infty$, and so $\| \tilde{\varphi} \|_{1,2} = 1$ - a contradiction as $\tilde{\varphi} \equiv 0$. □

We now have the following.

Proposition 7.3.7 *Let ΔE be of the form (7.3.18) and let $\alpha_n = \hat{\alpha}_n + \varepsilon\sigma_n$, $\beta_n = \hat{\beta}_n + \varepsilon\tau_n$, where $\hat{\alpha}_n = 0$ and $\hat{\beta}_n = 0$ ($n = 1, 2, 3, \ldots$). Then the second variation $(\Delta E)_{,\varepsilon\varepsilon}|_{\varepsilon=0}$ around $\hat{\alpha}_n = 0$ and $\hat{\beta}_n = 0$ is such that*

$$(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0} \geq \sum_{n=1}^{\infty} C_n' \|\sigma_n\|_{1,2}^2 + D_n' \|\tau_n\|_{1,2}^2,$$

with C_n', $D_n' > 0$ for all n.

Proof: As $(\Delta E)_{,\varepsilon\varepsilon}|_{\varepsilon=0}$ can be written in the form (7.3.20), the result follows from applying Proposition 7.3.6 to the term

$$\int_a^b A(R)((\sigma_n')^2 + (\tau_n')^2) + B(R)(\sigma_n\sigma_n' + \tau_n\tau_n') + C_n(R)(\sigma_n^2 + \tau_n^2)\ dR$$

for each value of n. □

7.4 Discussion of the general case.

We now consider the most general case where $\tilde{\boldsymbol{u}}$ is of the form (7.0.2) with $\tilde{\rho}(R,\theta)$ and $\tilde{\psi}(R,\theta)$ satisfying (G1) and (G2). In this case the Dirichlet integral is of the form (7.2.9). From Proposition 7.2.1, we can write

$$\begin{aligned}\int_{R=a}^b \int_{\theta=0}^{2\pi} R\left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 + \frac{1}{R}\left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial \theta}\right|^2 dR\, d\theta \\ \geq \int_{R=a}^b \int_{\theta=0}^{2\pi} R\left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 dR\, d\theta + 2\pi \int_a^b \frac{\rho^2(R)}{R}\, dR.\end{aligned}$$

The problem is to see if

$$\int_{R=a}^{b} \int_{\theta=0}^{2\pi} R \left| \frac{\partial \tilde{\boldsymbol{u}}}{\partial R} \right|^2 \, dR \, d\theta \geq 2\pi \int_{R=a}^{b} R(\rho'(R))^2 + R\rho^2(R)(\psi'(R))^2 \, dR$$

for suitably chosen $\rho(R)$ and $\psi(R)$. Now the function $\rho(R)$ is obtained using (7.2.10). However, it is not clear how to choose $\psi(R)$, and there is no simple symmetrisation as in §7.3.1 where $\tilde{\rho}(R,\theta) = \rho(R)$. The following two Fourier series approaches, similar to that used in §7.3.2 in the case $\tilde{\psi}(R,\theta) = \psi(R)$, have been considered but problems still remain.

(1) In the first, we write $\tilde{\rho}(R,\theta)$ in the form $\tilde{\rho}(R,\theta) = \rho(R) + \phi(R,\theta)$ where

$$\phi(R,\theta) = \sum_{n=0}^{\infty} \alpha_n(R)\cos(n\theta) + \beta_n(R)\sin(n\theta),$$

and $\tilde{\psi}(R,\theta)$ in the form $\tilde{\psi}(R,\theta) = \psi(R) + \sigma(R,\theta)$ where

$$\sigma(R,\theta) = \sum_{n=0}^{\infty} \gamma_n(R)\cos(n\theta) + \delta_n(R)\sin(n\theta).$$

Upon putting

$$\Delta E = \frac{1}{2}\int_A |\nabla \tilde{\boldsymbol{u}}|^2 \,\mathrm{dx} \; - \frac{1}{2}\int_A |\nabla \boldsymbol{u}|^2 \,\mathrm{dx}$$

an expression generalising (7.3.17) can be obtained for ΔE, after a lengthy calculation. However, it is an intractable expression and is cannot be presented as a sum and difference of squares similar to (7.3.13). Further, upon putting $\alpha_n = \hat{\alpha}_n + A$, $\beta_n = \hat{\beta}_n + B$, $\gamma_n = \hat{\gamma}_n + \Gamma$ and $\delta_n = \hat{\delta}_n + \Delta$ with $\hat{\alpha}_n$, $\hat{\beta}_n$, $\hat{\gamma}_n$, $\hat{\delta}_n \equiv 0$ for $n = 1, 2, 3, \ldots$, a complicated expression is obtained for the second variation $(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0}$, but it is not of a form similar to (7.3.19) and it is not clear in this generality that $(\Delta E)_{,\varepsilon\varepsilon}\big|_{\varepsilon=0}$ is positive definite.

(2) It is possible to consider a second Fourier series approach by writing

$$\tilde{\rho}(R,\theta) \begin{pmatrix} \cos(\theta + \tilde{\psi}(R,\theta)) \\ \sin(\theta + \tilde{\psi}(R,\theta)) \end{pmatrix}$$
$$= (\rho(R) + \tilde{\phi}(R,\theta)) \begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix} + \tilde{\sigma}(R,\theta) \begin{pmatrix} -\sin(\theta + \psi(R)) \\ \cos(\theta + \psi(R)) \end{pmatrix}$$

and then considering

$$\tilde{\phi}(R,\theta) = \sum_{n=0}^{\infty} \tilde{\alpha}_n(R)\cos(n\theta) + \tilde{\beta}_n(R)\sin(n\theta)$$

and

$$\tilde{\sigma}(R,\theta) = \sum_{n=0}^{\infty} \tilde{\gamma}_n(R)\cos(n\theta) + \tilde{\delta}_n(R)\sin(n\theta).$$

In this case the Dirichlet integral is in a simpler form than with the first Fourier series expansion considered in (1) above, and we can write the integral of the square of the radial component as

$$\begin{aligned}
&\frac{1}{2}\int_A \left|\frac{\partial \tilde{\boldsymbol{u}}}{\partial R}\right|^2 \,\mathrm{dx} \\
&= \pi \int_{R=a}^{b} R\Bigg\{\left(\left(\frac{d\rho}{dR}+\frac{d\alpha_0}{dR}\right) - \gamma_0\frac{d\psi}{dR}\right)^2 + \left(\frac{d\gamma_0}{dR} - (\rho+\alpha_0)\frac{d\psi}{dR}\right)^2 \\
&\qquad + \sum_{n=1}^{\infty}\Bigg[\left(\frac{d\alpha_n}{dR} - \gamma_n\frac{d\psi}{dR}\right)^2 + \left(\frac{d\beta_n}{dR} - \delta_n\frac{d\psi}{dR}\right)^2 \\
&\qquad\qquad + \left(\frac{d\gamma_n}{dR} + \alpha_n\frac{d\psi}{dR}\right)^2 + \left(\frac{d\delta_n}{dR} + \beta_n\frac{d\psi}{dR}\right)^2\Bigg]\Bigg\}\, dR.
\end{aligned} \tag{7.4.1}$$

The area constraint (7.2.10) is written as

$$\begin{aligned}
\alpha_0^2(R) + 2\rho(R)\alpha_0(R) + \gamma_0^2(R) =& \sum_{n=1}^{\infty} n(\beta_n(R)\gamma_n(R) - \alpha_n(R)\delta_n(R)) \\
&- \sum_{n=1}^{\infty}\frac{1}{2}(\alpha_n^2(R) + \beta_n^2(R) + \gamma_n^2(R) + \delta_n^2(R)).
\end{aligned} \tag{7.4.2}$$

We can write α_0 in terms of γ_0 and α_n, β_n, γ_n and δ_n ($n \in \mathbb{N}$). However, on substituting this expression for α_0 we cannot write (7.4.1) as a sum and difference of squares similar to (7.3.13) as we still have cross terms that are not eliminated. In particular, in the special case where $\tilde{\psi}(R,\theta) = \psi(R)$ all the terms in $\psi'(R)$ were eliminated by using the relation (7.3.12), whereas they are not eliminated here by using (7.4.2). This will cause problems, particularly as we still don't have a symmetrisation from $\tilde{\psi}(R,\theta)$ to $\psi(R)$.

Chapter 8

Concluding remarks.

Investigation of minimising properties of rotationally symmetric equilibria.

We recall that our aim in Chapter 7 was to show that minimisers of the total stored energy of the form

$$E(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}|^2 + h(\det(\nabla \boldsymbol{u}))\ \mathrm{dx} \tag{8.0.1}$$

in a class of general deformations of the form

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \tilde{\rho}(R,\theta) \begin{pmatrix} \cos(\theta + \tilde{\psi}(R,\theta)) \\ \sin(\theta + \tilde{\psi}(R,\theta)) \end{pmatrix} \tag{8.0.2}$$

satisfying symmetric displacement boundary conditions are necessarily rotationally symmetric. Our approach was to say that for every non-rotationally symmetric map there is a corresponding symmetrised map (related by the area constraint (7.0.3)) and to show that the symmetrised map necessarily had less energy. Partial results were obtained using this approach (Proposition 7.3.1 and Proposition 7.3.7).

However, the question of rotational symmetry of the minimisers is still unresolved. It may be the case that a different symmetrisation could yield stronger results, but it would not be clear as to how the term

$$\int_A h(\det(\nabla \tilde{\boldsymbol{u}}(\boldsymbol{x})))\ \mathrm{dx} \tag{8.0.3}$$

would be affected. The reason why an area-preserving symmetrisation (as in Definition 7.0.1) was chosen was because the term (8.0.3) was lowered by such a symmetrisation (see Proposition 7.1) and the problem reduced to that of lowering the Dirichlet integral.

Minimising properties of rotationally symmetric equilibria in slightly compressible elasticity.

In §6.3 it was shown that if an equilibrium solution $\bar{\boldsymbol{u}}_0$ is a weak local minimiser of the energy $E_0(\boldsymbol{u})$ given by

$$E_0(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}(\boldsymbol{x})|^2 \, \mathrm{dx}$$

for an incompressible annulus, then for $\delta \in (0, \bar{\delta})$ an equilibrium solution $\bar{\boldsymbol{u}}_\delta$ is a weak local minimiser of the energy $E_\delta(\boldsymbol{u})$ given by

$$E_\delta(\boldsymbol{u}) = \int_A \frac{1}{2}|\nabla \boldsymbol{u}(\boldsymbol{x})|^2 + \frac{1}{\delta} h(\det(\nabla \boldsymbol{u}(\boldsymbol{x}))) \, \mathrm{dx}$$

for a slightly compressible annulus (see Proposition 6.3.2). This was done by assuming that a family of equilibria $\bar{\boldsymbol{u}}_\delta$ converges to $\bar{\boldsymbol{u}}_0$ as the compressibility δ tends to zero. An interesting problem would be to assume that $\bar{\boldsymbol{u}}_0$ is rotationally symmetric (this is justified by Proposition 6.2.9) and to see if $\bar{\boldsymbol{u}}_\delta$ is necessarily rotationally symmetric in order for the result of Proposition 6.3.2 to hold, or if there is a counterexample where Proposition 6.3.2 holds for $\bar{\boldsymbol{u}}_0$ rotationally symmetric but $\bar{\boldsymbol{u}}_\delta$ not rotationally symmetric. A second problem would be to justify the expansion $\bar{\boldsymbol{u}}_\delta = \bar{\boldsymbol{u}}_0 + \delta \bar{\boldsymbol{u}}_1 + \ldots$ for rotationally symmetric solutions by use of the implicit function theorem.

Uniqueness of semi-inverse minimisers for an annulus.

In Chapter 4 the results of existence and regularity of rotationally symmetric minimisers of the form

$$\boldsymbol{u}(\boldsymbol{x}) = \rho(R) \begin{pmatrix} \cos(\theta + \psi(R)) \\ \sin(\theta + \psi(R)) \end{pmatrix} \tag{8.0.4}$$

given in [57] were extended to stored energy functions of the form

$$W(\nabla \boldsymbol{u}) = \tilde{g}\left(\left\{(\rho')^2 + \left(\frac{\rho}{R}\right)^2 + (\rho\psi')^2\right\}^{\frac{1}{2}}, \rho' \frac{\rho}{R}\right) \tag{8.0.5}$$

on the set

$$\mathcal{A}_N^{\mathrm{sym}} = \{(\rho,\psi) \in W^{1,1}((a,b)) : \rho(a) = a, \rho(b) = b, \rho'(R) > 0 \text{ a.e. on } (a,b),$$
$$\psi(a) = 0, \psi(b) = 2N\pi\}. \tag{8.0.6}$$

(See Proposition 4.1.2 and Proposition 4.1.3.) It is still an open question as to whether for each value of N there is uniqueness of minimisers in $\mathcal{A}_N^{\mathrm{sym}}$ for the total energy of the form

$$\begin{aligned} E(\boldsymbol{u}) & \\ &= 2\pi \int_a^b R\left\{\frac{1}{2}\left[(\rho'(R))^2 + \left(\frac{\rho(R)}{R}\right)^2 + (\rho(R)\psi'(R))^2\right] + h\left(\rho'(R)\frac{\rho(R)}{R}\right)\right\}\, dR \\ &:= 2\pi \int_a^b \Phi(R,(\rho(R),\psi(R)),(\rho'(R),\psi'(R)))\, dR. \end{aligned} \tag{8.0.7}$$

One possible approach is by field theory. For the case of cavitation of a unit ball, Sivaloganathan in [61] shows uniqueness of cavitating equilibria for boundary value problems in the case of radial deformations by such an approach. In [61] a field of extremals is constructed as a consequence of the invariance of the equilibrium equations under the one-parameter family of functions $\tilde{r}(R,\delta) := \delta r\left(\dfrac{R}{\delta}\right)$. For our problem we will need to consider a two-parameter family of functions and see whether this family of functions is a field of extremals. Now the Euler-Lagrange equations for the energy of the form (8.0.7) are

$$\begin{aligned} &\frac{d}{dR}\left[R\rho'(R) + \rho(R)h\left(\rho'(R)\frac{\rho(R)}{R}\right)\right] \\ &\qquad = \frac{\rho(R)}{R} + R\rho(R)(\psi'(R))^2 + \rho'(R)h\left(\rho'(R)\frac{\rho(R)}{R}\right) \end{aligned} \tag{8.0.8}$$

and

$$\frac{d}{dR}\left[R\rho^2(R)\psi'(R)\right] = 0 \tag{8.0.9}$$

for $R \in (a,b)$, and we note that (8.0.8) & (8.0.9) are invariant under the two-parameter transformation group

$$\Pi((\varepsilon,\delta),(R,\rho,\psi)) := (\varepsilon R, \varepsilon\rho, \psi + \delta)\,. \tag{8.0.10}$$

However, $\Pi((\varepsilon,\delta),(R,\rho,\psi))$ cannot result in a field of extremals since the Lagrange brackets $[\varepsilon,\delta]$ given by

$$[\varepsilon,\delta] := \frac{\partial\tilde{\rho}}{\partial\varepsilon}\frac{\partial}{\partial\delta}\left(\frac{\partial\Phi}{\partial\rho'}\right) - \frac{\partial\tilde{\rho}}{\partial\delta}\frac{\partial}{\partial\varepsilon}\left(\frac{\partial\Phi}{\partial\rho'}\right) + \frac{\partial\tilde{\psi}}{\partial\varepsilon}\frac{\partial}{\partial\delta}\left(\frac{\partial\Phi}{\partial\psi'}\right) - \frac{\partial\tilde{\psi}}{\partial\delta}\frac{\partial}{\partial\varepsilon}\left(\frac{\partial\Phi}{\partial\psi'}\right) \tag{8.0.11}$$

are not zero (the Lagrange brackets equalling zero being a necessary condition for the existence of a field of extremals (see, e.g., [33])). It is still possible to try and show uniqueness by field theory, but this would require a different two-parameter family of functions to (8.0.10); I have yet to find a suitable two-parameter family of functions that results in both the Euler-Lagrange equations (8.0.8) & (8.0.9) being invariant and the Lagrange brackets (8.0.11) being zero.

For boundary value problems in finite elasticity, Valent [70] has shown uniqueness of solutions $\bar{\boldsymbol{u}}$ of the equilibrium equations

$$\operatorname{div}(\boldsymbol{T}(\nabla\bar{\boldsymbol{u}})) + \boldsymbol{f} = \boldsymbol{0}$$

close to the identity using the implicit function theorem, under identity boundary conditions. Zhang [72] uses these results in the case of polyconvex stored energy functions $W(F)$ to show uniqueness of solutions $\bar{\boldsymbol{u}}$ of the equilibrium equations

$$\operatorname{div}\left(\frac{\partial W}{\partial F}(\nabla\bar{\boldsymbol{u}})\right) + \boldsymbol{f} = \boldsymbol{0}$$

close to the identity for pure displacement boundary value problems, under the assumption of strong ellipticity (see [72, §2]). Zhang also shows that the solution $\bar{\boldsymbol{u}}$ obtained by Valent's method using the implicit function theorem is the same as the minimiser of the energy functional

$$E(\boldsymbol{u}) = \int_\Omega W(\nabla\boldsymbol{u}) - \boldsymbol{f}\cdot\boldsymbol{u}\,\mathrm{dx}$$

derived by Ball using the direct method of the calculus of variations (see [72, §3]). For the case of an annulus and for polyconvex stored energy functions, since the identity map is a unique global minimiser, it is possible to apply the results of Valent [70] and Zhang [72] in order to show the existence of a unique minimiser for a small prescribed twist of the boundary. The question is whether this approach can be extended to maintain uniqueness up to a twist of 2π of the outer boundary. The computational results obtained using AUTO (as shown in §4.1) suggest that

there is no bifurcation in the class of symmetric maps as the outer boundary is twisted. The results of §6.3 also suggest that bifurcation does not occur for slightly compressible materials.
We briefly remark that in [62] Sivaloganathan has shown that a solution of the Euler-Lagrange equations is a strong local minimiser in the case where the stored energy function is polyconvex and the variations have small support and he has pointed out that given any particular equilibrium solution the techniques of [62] could be used to produce stronger results. However, in the case of a compressible annulus the problem of showing that a symmetric equilibrium solution is even a *weak* local minimiser is still open.

The use of frame-indifference and isotropy of $W(F)$ in proving the results of §4.1.

We showed in Chapter 4 that a solution of the rotationally symmetric Euler-Lagrange equations (8.0.8) & (8.0.9) gives rise to a solution of the full equilibrium equations for stored energy functions of the form (8.0.5) (see Proposition 4.1.5). It would be interesting to try and derive the result of Proposition 4.1.5 using only the assumptions of frame-indifference and isotropy of the stored energy function.

Existence and regularity of semi-inverse minimisers in three dimensions.

It is open as to whether there is a three-dimensional analogue of the results of Propositions 4.1.2 - 4.1.5. One possibility is to consider the example in [57]. There the three-dimensional example is that of a "hollow tube" of radius c and thickness $(b-a)$. Thus the region that the nonlinear elastic material occupies is a torus with annular cross-section:

$$\Omega = \{(x,y,z) \in \mathbb{R}^3 : a^2 \leq [(x^2+y^2)^{\frac{1}{2}} - c]^2 + z^2 \leq b^2\}, \tag{8.0.12}$$

where a, b, $c \in \mathbb{R}$ are constants with $0 < a < c < b$. There are still open questions as to what is an appropriate class of semi-inverse deformations and whether any such class will generate semi-inverse minimisers which are also solutions of the full three-dimensional equilibrium equations.

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